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Asset Pricing and Robust Control

ΣΩΤΗΡΙΟΣ ΛΕΒΑΚΟΣ

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Περίληψη

Σκοπός της διατριβής αυτής είναι να εξετάσει εφαρμογές του robustcontrol, ως εργαλείο μοντελοποίησης της αβεβαιότητας (uncertainty) - όπως αυτή διαφοροποιείται από τον κίνδυνο (risk) -σε ορισμένες πτυχές της οικονομικής επιστήμης. Η πρώτη σαφής διάκριση της αβεβαιότητας από τον κίνδυνο οφείλεται στον Knight (1921), ο οποίος διαχώρισε τις περιπτώσεις όπου αποφάσεις λαμβάνονται υπό την πλήρη και σαφή γνώση των πιθανοτήτων, από τις καταστάσεις όπου οι πιθανότητες δεν είναι γνωστές ή είναι αμφισβητήσιμες. Όπως απέδειξε ο Ellsberg (1961),η αποστροφή στην αβεβαιότητα οδηγεί σε τελείως διαφορετικές συνέπειες από την αποστροφή στον κίνδυνο, ενώ έρχεται σε αντίθεση με την παραδοσιακή θεωρία χρησιμότητας. Ενώ η επίδραση του κινδύνου στην λήψη αποφάσεων στην οικονομική επιστήμη έχει μοντελοποιηθεί και μελετηθεί εκτενέστατα στο παρελθόν,η επίδραση της αβεβαιότητας ξεκίνησε να εξετάζεται μόλις πρόσφατα. Ένα πολύ μεγάλο μέρος της βιβλιογραφίας βασίζεται στις μεθοδολογίες του RobustControlTheory- κλάδος της μηχανικής και των εφαρμοσμένων μαθηματικών - προκειμένου να μοντελοποιήσει την λήψη αποφάσεων κάτω από συνθήκες αβεβαιότητας. Στην διατριβή αυτή εξετάζονται εφαρμογές στην αποτίμηση αξιογράφων(assetpricing) καθώς και στην σύνθεση χαρτοφυλακίων. Η αβεβαιότητα φαίνεται να δημιουργεί μια πιο επιφυλακτική συμπεριφορά, η οποία ενσωματώνεται στις τιμές των αξιογράφων. Βασιζόμενοι στον συγκεκριμένο τρόπο μοντελοποίησης της αβεβαιότητας, διάφοροι συγγραφείς υποστηρίζουν ότι το επασφάλιστρο κινδύνου που παρατηρείται στην αγορά (marketpriceofrisk), στην πραγματικότητα αποζημιώνει τόσο για τον κίνδυνο όσο και για την αβεβαιότητα. Τέλος, παρουσιάζεται η συμβολή του RobustControlστην δημιουργία χαρτοφυλακίων με μειωμένη ευαισθησία σε τυχόν αποκλίσεις των παραμέτρων των συστατικών αξιογράφων. Η απόδοση τέτοιων χαρτοφυλακίων επηρεάζεται πολύ λιγότερο από λανθασμένες εκτιμήσεις των στατιστικών χαρακτηριστικών των αξιογράφων.

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Chapter 1: Introduction

1.1 Risk and Uncertaintyin Financial Economics

The first clear distinction of uncertainty versus risk in finance is attributed to Knight (1921) who in his classic work "Risk, Uncertainty, and Profit", characterizes risk as "randomness that can be measured precisely" by assigning numerical probabilities. On the other hand, in situations where:

- the decision maker is ignorant of the statistical frequencies of events relevant to his decisions, or
- a priori calculations are impossible, or
- the relevant events are in some sense unique, or
- an important once and for all decision was concerned,

numerical probabilities are inapplicable and the decision maker faces "uncertainty". According to Knight, this difference is very important to markets, since risk is insurable through exchange, while uncertainty is not: well-organized institutions are able to price and market insurance contracts that only depend on risky phenomena; uncertainty, however, creates frictions that institutions may not be able to accommodate.

In his book, "The Foundations of Economics" (1954), Savage proposed the subjective expected utility model, a theory of decision making under uncertainty, which until now remains the standard model of choice in financial economics. Based on Bayesian probability theory and by combining two subjective concepts, the personal utility function and the personal probability distribution, Savage proposes seven axioms of choice that **a rational individualought to follow**. According to that, individuals assign a priori probabilities expressing their personal beliefs, to possible outcomes of an uncertain event and given the personal utility of each outcome they formulate the subjective expected utility of the event. Which decision an individual would actually prefer, depends on which subjective probability is higher. Different people make different decisions because they might have different utility functions or different beliefs about the probabilities of different outcomes. However, it has repeatedly been shown in experimental settings that decision makers do not exhibit this kind of behavior.

Ellsberg (1961) challenges the Savage Bayesian model and argues that individuals display choice patterns that are inconsistent with the existence of beliefs representable by a probability measure. Using simple though experiments, he demonstrated that decision-makers prefer environments with known odds to those with uncertain probabilities. He suggested the following mind experiment that is known as the Ellsberg Paradox: Subject is presented with two urns, each containing 100 balls, either red or black. Urn A contains 50 red balls and 50 black ones, while the composition of urn B is unknown. Before drawing a ball from each urn, subject is asked to place a bet on the color of the ball drawn from **one** urn. It has been observed empirically that people prefer to bet on color of the ball drawn from urn A rather than placing a bet on the color of the ball drawn from urn B. Such behavior, verified by empirical observations, is inconsistent with the subjective expected utility model, which predicts either indifference between urns, or a preference for Urn B depending on the prior probability assignment.

Gilbert and Schmeidler (1989) offer the following conceivable explanation to the experiment: subject has too little information to form a prior probability distribution over the composition of the urn B. She considers the worst case scenario where urn B doesn't contain any balls of the color of her choice and since she has no chances of winning, her expected utility is zero. Comparing with urn A, where the probability distribution is known and the chance of winning is 1/2, betting on urn A has a positive expected utility. Subject maximizes her utility by betting on urn A. This behavior is consistent with Ellsberg axioms.

During the past two decades, a considerable amount of literature in financial economicshas been devoted on modeling Knightian uncertainty and extending the existing workon intertemporal preferences, decision-making (portfolio and consumption rules) and asset pricing. There are several good reasons for this:

• Ellsberg paradox demonstrated that risk and uncertainty are distinct characteristics that imply very different behavior in random environments. The expected utility theory that was until recently the cornerstone of financial modelingdoes not take uncertainty into account, and therefore does not induce uncertainty related preferences. Furthermore, expected utility fails to explain behavior demonstrated by individuals in stylized experiments. On the other hand, Ellsberg's axioms are holding up in experimental settings.

• Decision makers often doubt the model at hand and consider it only as an approximation of the "real one". They believe that actual data will come from an unknown member of a set of unspecified models near the approximating one and therefore adjust the decision rules as to protect themselves against modeling and specification errors. Econometricians also usually face a model detection problem; having at their disposal only finite observational data, they cannot distinguish between alternative models, as data would fit any of them. This behavior is in sharp contrast with rational expectations, where agents simply know the true model and trust it.

• Finally, many topics in financials economics remain open and cannot be explained with existing theory. One such famous example is the equity premium puzzle, a term first introduced by Mehra and Prescott (1985). Asset returns observed in financial markets bear a premium (compared to the risk-free rate) that is an order of magnitude greater than what theory would predict for compensating for bearing risk alone. Modifying the assumed preferences of investors as to incorporate uncertainty aversion would offer a possible explanation to this and other puzzles.

There are two prominent approachesforincorporating uncertainty into financial economics. A large part of literature is inspired from the robust control literature, a branch of control theory in engineering and applied mathematics. The second approach is based on the work of Gilboa and Schmeidler (1989) whoaxiomatized preferences under uncertainty and proposed max-min expected utility decision making. Other promising methods of modeling preferences under uncertainty include the "smooth ambiguity" model of Klibanoff et al. (2005) and scale invariant ambiguity-averse preferences, Skiadas (2011).

1.2 Modeling Uncertainty and Robust Control

Many of the ideas and inspiration in economics come from the control theory, an interdisciplinary field of science originating from engineering and mathematics. Optimal control theory, a mathematical method for solving dynamic optimization problems, remains widely used even today in the intertemporal allocation of scarce resources and decision making in financial economics, macroeconomics and resource economics.Optimal control deals with the problem of finding a control law for a given *system*, such that an

intertemporal objective is optimized (or a certain optimality criterion is achieved). The system is described by a mathematical model, usually a set of differential equations which specify how the state of the system evolves, influenced also by the control of the decision maker. Althoughoptimal control gained a significant foothold in engineering mainly due to the work of Kalman in linear quadratic control and filtering in the 1970s, it was not long before it showed its limitations under model misspecification: the model assumed to describe the system is not correct or un-modeled disturbances show up. Minor differences in model parameters or system disturbances can lead to sub-optimal decision rules or even system instability, which in engineering problems can be catastrophic. Robust control theory emerged in the 1980s (and is still active today) to explicitly deal with model uncertainty and address a major concern in engineering, the desire for system stability. Robust control rules are designed to guarantee a given level of performance as long as the uncertain parameters of the model and the disturbances remain within agiven set. The worst case philosophy was adopted out of concerns for stability; the decision rule has to meet and maintain specific performance criteria in all cases, and especially under the worst case scenario. Thus, a lower bound on performance is achieved. One of the most important techniques that was employed to achieve this, is the min-max approach.

The designer of the robust rules starts with a nominal model of the system, a objective function that is the subject of maximization (in general optimization) and a set of performance criteria that have to be achieved. He considers the size of model misspecification he wants to guard against by formulating a set of alternative models; the robust rule must meet the performance criteria under all the models of this set. The set of alternative models is usually obtained by bounding the size of perturbations that the designer considers. Then the solution to the control problem is obtained by a two player dynamic game implementation: the designer maximizes the objective function, while a fictional malevolent second agent is minimizing it by his choice of control. The control of the malevolent agent is constrained by the set of alternative models (the size of perturbations). Since the objective function is maximized under the worst case alternative model (worst case control of the malevolent agent) a lower bound on performance is achieved. Positing a malevolent agent is just a device that the decision maker uses to perform a systematic analysis of the fragility of alternative decision rules and to construct a lower bound on the performance it can be attained by using them. A designer who is concerned about

robustness naturally seeks to construct bounds on the performance of potential decision rules and then malevolent agent helps the decision maker do that.

In most applications in economics, choices are modeled through a decision maker. This typically involves a utility function (expressing the objective) that is maximized subject to a model that describes the environment. Concerns about uncertainty can be represented by altering the utility function in a specific way and/or distorting the decision maker's expectations relative to the model at hand in a context specific way. Robust control is used to model preferences under uncertainty by distorting how expectations for future periods are formed and thereby by altering decisions.

The most influential approach to robust control in economics is owed to Hansen and Sargent and their co-authors, summarized in their monograph (2007), where they provide a framework for setting up and solving robust control problems with applications in finance and macroeconomics. Hansen and Sargent explicitly deal with the issue of defining the class of alternative models that the decision makers guards against by introducing the concept of relative entropy (or Kullback-Liebler distance) to measure and bound the distance of the model uncertainty set from the nominal model. This is based on the difficulty to distinguish between two models, given a finite number of data. Statistical model detection theory is used to calculate the entropy penalty, representing the strength of the decision maker's preference for robustness. This approach is applied to three different types of economic environments, expanding the concept of equilibrium under rational expectations: a competitive equilibrium with complete markets in history-contingent claims and a representative agent who fears model misspecification; a Markov perfect equilibrium of a dynamic game with multiple decision makers who fear model misspecification; and a Stackelberg or Ramsey problem in which the leader fears model misspecification. Some other contributions include the formulation of discounted problems (robust literature treats undiscounted problems) and the multiple-agent setting.

Hansen and Sargent (1999) derive equilibrium asset prices and define the market price of uncertainty by considering an investor who faces uncertainty about the state of the economy in a permanent income model. Uncertainty aversion generates a premium additional to the risk premium that is priced. Anderson et al (2003) extend the above framework in continuous time and link the market price of uncertainty with the difficulty the

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agent faces to distinguish between possible models. However, uncertainty alone cannot explain the equity premium puzzle. Maenhout (2004) extends the framework of Anderson et al. (2003) by including homothetic preferences and clearly distinguishing three different kinds of preferences: risk aversion, uncertainty aversion and intertemporal substitution. One key conclusion is that uncertainty aversion cannot be captured by stylized experiments with known probabilities and is only present in specific context environments. Vardas and Xepapadeas (2004) derive robust portfolio choices when the investor faces uncertainty about the statistics of the risky asset(s) in aiCAPM setting. Maccheroni et al. (2006) introduce and axiomatize dynamic variational preferences by generalizing the multiple priors model of Gilboa& Schmeidler (1989) by including the robust control approach and the mean variance preferences of Markowitz & Tobin. Barillas, et al. (2007) show that market prices of model uncertainty contain information about compensation for removing model uncertainty and not consumption. Kleshchelskiand Vincent (2008) show that uncertainty aversion greatly amplifies the effect of stochastic volatility in consumption growth and therefore can explain asset prices in an empirically plausible way. Hansen & Sargent (2011) develop statistical measures to calibrate the decision maker's uncertainty aversion. Statistical model detection calculations are used to calculate entropy penalties, representing preferences expressing model ambiguity and concerns about statistical robustness. Finally, Epstein & Schneider (2010) provide a unifying framework for considering the multiple priors model, the robustcontrol inspired models and the smooth ambiguity model. It is shown that all three models imply very different behavior from the subjective utility theory leading to different results in asset pricing and portfolio choice. A common theme is that ambiguity averse agents choose more conservative positions, and, in equilibrium command additional "ambiguity premia" on uncertain events.

Max-min approach has been accused of being "too cautious", because the decision maker puts too much weight on a "very unlikely" scenario. However, most approaches in the robust control inspired literature (like in Hansen and Sargent) consists of carefully calibrating the model perturbations that are considered so that the model uncertainty set is difficult to distinguish from the approximating model. This way, the worst case model fits the available data almost as well as the approximating model. Moreover, by inspecting the implied worstcase model, it can easily be evaluated whether the decision maker is focusing on scenarios that appear to be too extreme. There are some approaches which seek a middle ground between the average case and the worst case (Tornell, 2003), however they have been less prominent.

The approach discussed above uses unstructured uncertainty model sets – the perturbations of the model are bounded but have no particular form, however there are good reasons for the decision maker to want put structure in the uncertainty set. The decision maker may be more confident for some aspects of the model relative to others or have a discrete set of models in mind. Not taking into account the particular structure may give a misleading impression of the actual uncertainty the decision maker faces. Some applications of robust control theory mainly in macroeconomics attempts to put structure to the uncertainty set. Levin and J. Williams (2003), Cogley and Sargent (2005) and Svensson and Williams (2006) focus on uncertainty sets with discrete possible models. Onatski& Stock (2002) and Onatski& Williams (2003) focus on different parametric model specifications, including uncertainty about dynamics (lags and leads), variables which may enter, uncertainty about data quality, and other futures which are built into parametric extensions of the nominal model. Most importantly, Onatski& Williams (2003) develop an example showing that the Hansen & Sargent approach may lead to the design of robust policy rules that can be destabilized by small parametric perturbations. While the robust rule may resist shocks of a certain size, small variations in the underlying model can result in disastrous policy performance. Thus, the particular structure and measurement of uncertainty can have important applications for decisions. Although there are important stability and performance criteria, constructing control rules for structured uncertainty is a more demanding task and the theory is not as fully developed as in the unstructured case.

Finally, another approach in modeling uncertainty aversion is owed to Gilboa and Schmeidler (1989) who proposed max-min expected utility decision rules using multiple priors, and axiomatized preferences under uncertainty. Under this model, an individual is considering the set of all possible probability distributions over the possible outcomes of the uncertain event. Being uncertainty averse, she maximizes her utility with respect to the least favorable set. Some applications of max-min expected utility include Epstein and Wang (1994) who explore equilibrium asset prices, Rigotti and Shannon (2004) who argue that under uncertainty some assets are not traded, Gagliardini et al. (2007) who discuss the uncertainty premium and term structure of interest rates and many others. More recently, Epstein and Schneider (2003) have extended the atemporal, static environment of Gilboaand

Schmeidler to an intertemporal dynamic context. Although the models based on multiple prior expected utility have been criticized of implying extreme behavior, it is widely used in the relevant literature.

1.3 Robust Control and Asset Pricing

Hansen et al. (1999) study consumption and saving profiles and market prices in the presence of model uncertainty. Their framework consists of a permanent income model with habit persistence as in Hall (1978). Agents are facing a planning problem –in the context of a discrete time, linear quadratic optimal control problem, they derive decision rules bymaximizing their expected lifetime utility.Expectations of future states lies in the model assumed. However, being uncertainty averse, they fear that future states will not evolve according to the model they used to form their expectations, but rather according to a model indistinguishably close to the one they have at hand. In other words, they suspect specification errors and want decisions to be insensitive to them. As in textbook robust control, Hansen et al. pose this as a max-min optimal control problem.Preference for robustness is introduced through a malevolent agent who is trying to minimize the lifetime utility of the agent through shifts in the conditional mean of the state vector. The control of the malevolent agent is bounded by the size of misspecification t agent wants to guard against. Decision rules are derived under the worst-case scenario, as to achieve a minimum level of performance. A very important result is then obtained. Formulas for consumption and assumptions rules are identical to ones coming from the usual permanent income models. Quantity allocations (consumption and saving plans) are observationally equivalent to preferences without robustness. To deduce asset prices, Hansen et al. consider an exchange economy in the style of Lucas (1978) and Epstein (1988). A large number of identical agents trade in security markets. Consumption and investment processes are equilibrium allocations for a competitive equilibrium; asset prices are obtained by finding shadow prices that clear security markets in equilibrium. Hansen et al. propose an extension to the Consumption Based Asset Pricing Model that is used in their setting. The stochastic discount factor is adjusted as to include a multiplicative component that reflects the representative agent's aversion to model uncertainty. It is a measure of "doubt" about the approximating model that the agent uses to evaluate future expectations. Hansen et al. heuristically define this multiplicative factor as the market price of Knightian uncertainty. There is a tight relationship between the market price of uncertainty and the probability of distinguishing the representative agent's approximating model from the worst case model. Mathematically, the multiplicative component is equivalent to the expected likelihood ratio of the "worst case model" relative to the "approximating model". For empirically plausible parameterizations of model uncertainty, the multiplicative component poses substantial variability, raising the theoretical value of equity premium and putting model uncertainty premia into market prices. The equity premium of asset returns compensates for bearing risk and model uncertainty, helping explain the equity premium puzzle.

Anderson et al. (2003) extend the discrete time linear control setting of Hansen et al. (1999) to a more general class of continuous time control problems in which the stochastic evolution of the state of the economy is a Markov diffusion process. Uncertainty aversionis introduced through concerns of model misspecification. The agent suspects that the diffusion process is not "true" and wants decision rules that work well when data conform to models that are statistically difficult to distinguish from the approximating model. The diffusion process is distorted by including a shift in the mean of the state process, leaving volatility unchanged. The size of the drift is bounded by robustness parameter θ , measuring the preference of robustness by defining the size of potential model misspecifications and indexing the set of alternative models considered. Decision rules are derived by maximizing the lifetimeutility of the decision maker (posed as a two period problem through a value function) under the worst case diffusion process.By adjusting robustness parameter θ , the worst case model can be designed to be close to the approximating model in the sense that is difficult to discriminate it from the original approximating model. To compute equilibrium prices, Anderson et al. follow Lucas (1978) and propose a similar modification to the consumption based asset pricing model as in Hansen et al. (1999). The stochastic discount factor generating the return premium is adjusted as to include a multiplicative component that generates model uncertainty premia in asset returns, reflectingthe representative agent's aversion to model uncertainty. A precise link is established between the uncertainty component of risk prices and the probability of distinguishing the decision maker's approximating and worst-case models(formalized by detection error probabilities); model uncertainty premium is bounded by robustness parameter θ that was used to define the worst case model. As a final step, the preference for robustness and model uncertainty premiumis calibrated to plausible values of detection error probabilities. The main conclusion is that aversion to model misspecification can account for a substantial but not all of estimated equity premium. The remaining fraction is impervious to details of model specification and is subject to alternative explanations (for example market frictions).

Maenhout (2004) considers an investor who is trying to decide how much to consume and how to allocate his savings between a risky and a riskless asset in a dynamic environment. For various good reasons, the investor worries about the expected return of the risky asset; although point estimates for asset return parameters are known and fixed, the investor has reasons to believe that the expected returnof the risky asset will notmaterialize. In other words, the investor faces uncertainty about the return process of the risky asset and wants decision rules that work reasonably well under the worst case scenario. The preference for robustness is modeled as a max-min robust control problem; the investor is trying to maximize his expected lifetime utility, while a malevolent agent, through a distortion in the mean of the asset's return process is minimizing it. The malevolent agent's control is constrained by an entropy penalty denoting the strength of the preference for robustness. Maenhout extents Anderson et al. (2003) by modifying the entropy penalty as to induce homothetic preferences. Robustness parameter θ still bounds the worst case considered and is calibrated using detection error probabilities. Decision rules are then obtained in closed form. Comparing with Merton's iCAPM (1971), consumption is not affected by the introduction of robustness, however, the optimal share of wealth allocated to the risky asset is greatly reduced. The results bear close resemblance to stochastic differential utility (SDU) the continuous time analogous of recursive utility. An investor with a homothetic preference for robustness and coefficient of risk aversion γ is observationally equivalent to an investor with SDU with effective risk aversion $\gamma + \theta$. What robustness does is to make the agent less willing to substitute across states as "effective" risk aversion increases, without changing the willingness to substitute intertemporally. To distinguish between three different types of behavior, risk aversion, uncertainty aversion and intertemporal substitution, Maenhout considers a decision maker with stochastic differential utility and robust homothetic preferences. Asset prices and the risk free rate in equilibrium are obtained in a Lucas style (1978) endowment economy. The equilibrium equity premium is given by a C-CAPM result where the price of risk is given by $\gamma + \theta$. Both market risk and model uncertainty are priced in equilibrium. Robustness also drives down the risk free rate through precautionary savings. Maenhout then makes a very important observation. By trying to measure "risk aversion" from empirical data, one would simply obtain the combined effect of risk aversion γ and uncertainty aversion θ or simply $\gamma + \theta$. Stylized experiments that try to estimate "risk

aversion" of individuals, usually report relative small values of γ because they involve situations where events are well specified and usually this environment-specific preference for robustness is not exhibited.Uncertainty aversion is environment specific, and although risk aversion γ might be constant across environments, difficult situations might generate a higher perceived $\gamma + \theta$.As a final step, Maenhout calibrates the equilibrium model by using data from the US economy. Given the observed risk free rate and excess return of the risky asset from two different time periods, the preference parameters (time, risk, uncertainty and intertemporal substitution) are adjusted as to satisfy the equilibrium model. Robustness parameter θ is also calibrated using error detection probabilities. Robustness helps resolve both the risk-free rate and the equity premium puzzle, as relatively reasonable parameter values reconcile the observational data.

Hansen et al. (2006) formally establish the link between robust control and the max-min expected utility of Gilboa and Schmeidler (1989). The cloud of models considered in robust control can be thought (and regarded) as a particular specification of Gilboa and Schmeidler's set of priors. However, none of the priors has the special status that the approximating model has in robust control theory. This poses a practical inconvenience, since an applied economist modeling with max-min expected utility would have to impute a set of models to the decision makers. On the other hand, an economist employing robust control would take a single approximating model and from it manufacture a set of models that express the decision maker's ambiguity. The link between these two lines of modeling is also discussed in Maccheroni et al. (2006a,b), Cerreia et al. (2008), and Strzalecki (2008).

1.4 Robust Control and Mean Variance Efficient Frontiers

Another prominent area of application of robust control in financial economics is the formulation of robust mean variance frontiers. Portfolio selection (originally formulated by Markowitz (1952)) is the problem of allocating capital over a number of available assets as to maximize return on the investment while minimizing risk. The "return" on a portfolio is measured by the expected value of the random portfolio return, and "risk" is quantified by the variance of the portfolio return. As there is in general a positive relationship between expected return and risk, both objectives cannot be achieved simultaneously. Given the maximum risk that an investor is willing to tolerate, the optimal allocation can be

obtained by solving an *optimization* problem. The set of all such Pareto optimal portfolios (in terms of return and risk) derived by this process is called the mean variance frontier.

Despite its theoretical success, the Markowitz Portfolio Theory has received a lot of criticism from practitioners. One of the key reasons for this is the model's susceptibility to parameter estimation errors. Market parameters, (like the mean return of an asset), are notoriously hard to estimate and subject to statistical errors. However, the optimization problem in the heart of the mean variance efficient frontier is very sensitive to perturbations in the parameters of the model. The results of the optimization are not very reliable as the performance of the portfolio could greatly vary for small parameter perturbations. The following quote of Michaud (1998) is indicative: "Although Markowitz efficiency is a convenient and useful theoretical framework for portfolio optimality, in practice it is an error-prone procedure that often results in error-maximized and investment-irrelevant portfolios".

A large part of the literature related to the problem of reducing the sensitivity to parameter fluctuations and designing robust mean variance efficient frontiers is inspired from robust control as it can explicitly deal with *uncertain* parameters and model misspecification. A common setting is that the investor worries and wants to guard against the worst case scenario, therefore, he seeks to formulate a portfolio that performs relatively well under all circumstances, as long as the uncertain parameters remain within specific bounds. This can be posed as a textbook robust control problem where the investor is trying to maximize the performance of the portfolio while a second malevolent agent is working on the opposite direction. The control of the malevolent agent is bounded by the set of uncertainty the investor considers and wants to guard against.

Kim and Boyd (2007) and (2008) define portfolio preferences under uncertainty as in Gilboa and Schmeidler (1989); an investorworrying about parameter uncertainty would consider all possible outcomes. Being uncertainty averse, he would choose the portfolio that performs best under the worst-case scenario. Under this formulation, the robust efficient frontieris defined as the set of all Pareto optimal portfolios under the worst-case scenario. The robust efficient frontier is derived by the robust control formulation discussed above. Kim and Boyd also present a numerical example to make some interesting observations: Robust efficient frontier portfolios do not perform as well as the nominal ones under the baseline scenario, i.e. if the point estimates of mean and variance hold exactly. However, they are less sensitive to variations in the parameters and achieve better performance than the nominal ones under the worst-case scenario. By choosing a portfolio of the robust efficient frontier, the investor is willing to trade a little part of baseline optimality to make sure that the performance of his portfolio is less sensitive to parameter variations and performs relatively better under the worst-case scenario. Finally, it appears that robust efficient portfolios are much more diversified than nominal ones (at the same risk levels), and therefore less likely to produce extreme results.

1.5 Outline

In chapters 2 and 3, two pivotal papers owed to Hansen and Sargent (1999) and Maenhout (2004) are summarized and discussed. Aquick introduction to the consumption based asset pricing model is also provided. In chapter 4 explores the application of robust control to mean variance efficient frontiers. Finally, chapter 5 summarizes.

Chapter 2: Consumption Based Asset Pricing Model with Uncertainty 2.1 The Consumption-Based Asset Pricing Model

A decision maker is faced with a simple dilemma: how much to consume today, how much to save for tomorrow, and what portfolio of assets to hold. Choice is driven by the fundamental desire for more *consumption* (rather than an intermediate objective such as the mean and variance of portfolio returns) and is modeled by Standard Expected Utility Theory, i.e. Von Neumann-Morgestern utility function:

$$U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})]$$
(2.1)

where c_t denotes consumption at time t, E_t is the conditional expectation on information available at time t, and β is the subjective discount factor that captures the decision maker's impatience. Utility function is increasing as to reflect desire for more consumption, and concave, reflecting the decreasing added value of each unit of additional consumption.

Consider an asset that has a price p_t at time t and pays a dividend d_{t+1} at time t+1. Its payoff next period is the new price plus the dividend: $x_{t+1} = d_{t+1} + p_{t+1}$. Note that x_{t+1} is a random variable. The investor cannot evaluate the exact value in a deterministic way, but can assess the probability distribution of various possible outcomes. The basic objective is to figure out the present value of the payoff by asking what is worth to the decision maker. Assume that the asset is freely traded and the investor can buy or sell a small quantity. His objective then is to maximize his utility:

$$\max_{\xi} u(c_{t}) + \beta E_{t}[u(c_{t+1})] s.t.$$

$$c_{t} = e_{t} - p_{t}\xi$$

$$c_{t+1} = e_{t+1} + \chi_{t+1}\xi$$
(2.2)

where ξ is the amount he chooses to buy and e_t the consumption level if he bought none of the asset. Solving for an optimal consumption and portfolio choice, the first order condition is obtained:

$$p_t u'(c_t) = E_t[\beta u'(c_{t+1})x_{t+1}]$$
(2.3)

The first terms expresses the marginal utility loss of consuming a little less today and buying a little more of the asset while the second term the expected utility gain from the asset's payoff in the future. Equation simply expresses that the decision maker simply wants to buy or sell the asset until the marginal loss equal the marginal gain. Re-arranging we obtain:

$$p_t = E_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right]$$
(2.4)

Using the investor's marginal utility to discount the payoff, the asset's price is simply the expected discounted value of the asset's payoff. Let us define by m_{t+1} the stochastic discount factor:

$$m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)}$$
(2.5)

 m_{t+1} is stochastic because it is not known with certainty at time *t*. It is often called *the* marginal rate of substitution, because it expresses the rate at which the decision maker is willing to substitute consumption at time *t*+1 for consumption at time *t*. The pricing formula then becomes:

$$p_t = E_t[m_{t+1}x_{t+1}] \tag{2.6}$$

This is by far the most important equation in modern asset pricing theory. It provides a most useful and insightful separation: one can incorporate all risk corrections by defining a single discount factor. Although we derived equation (2.6) using a utility function, it is still valid if we define the stochastic discount factor as:

$$m_{t+1} = \beta \frac{M_{t+1}^c}{M_t^c} \tag{2.7}$$

where M_{t+1}^c is the marginal utility gain from consumption. The discount factor can be thought as a random variable that generates prices from payoffs. Using the *law of one price* and the *absence of arbitrage* theorems, and without implicitly assuming all the structure of the investors, utility functions, complete markets and so forth, it can be proven that one such discount factor always exists (Cochrane, 2005). Indeed, by writing $p_t = E_t[m_{t+1}x_{t+1}]$ none of the below is assumed:

- Market completeness or a representative investor
- Market is in equilibrium or investors have bought all the securities they want to

- Normal distribution of asset returns and independence over time
- Two-period setting, quadratic utility or separable utility
- Absence of human capital or labor income.

Indeed, the only assumption that was made is that the investor can buy or sell a small quantity of the asset. Equation $p_t = E_t[m_{t+1}x_{t+1}]$ expresses a private valuation of the asset to the investor. If the private valuation of the asset is higher than the market value of the asset, the investor will continue to buy the asset as long as he can, and as long his personal valuation is higher than the market price. Some of above the assumption come only later, to derive market behavior.

All asset pricing models (CAPM, APT, iCAPM, etc.) are specializations of the basic pricing equation, simply using alternative ways of connecting the discount factor m_{t+1} to data. Marginal utility growth is not the easiest thing to measure from real life data; therefore, factor pricing models look for variables that are good proxies, like return on stock market indices, GDP growth, interest rates, or other macroeconomic variables, usually imposing additional assumptions:

$$m_{t+1} = f(data) \tag{2.8}$$

CAPM can be derived as a specialization of the basic pricing equation. By imposing additional assumptions (like normal asset return distribution or quadratic utility function, no labor income etc.), the discount factor m_{t+1} is *tied* to the return on the "wealth" portfolio, which is usually proxied by the return on a broad-based stock portfolio such as the value-weighted NYSE, S&P500 etc.:

$$m_{t+1} = a + bR_{t+1}^W (2.9)$$

The most important special case of the basic asset price equation is the return equation. The decision maker pays one consumption unit today to get *R* consumption units tomorrow. The pricing equation then becomes:

$$1 = E_t[m_{t+1}R_{t+1}] \tag{2.10}$$

Consider the case where a risk-free security is traded and the risk-free rate R^{f} is known ahead of time. In that case:

$$1 = E_t [m_{t+1} R^f] \Rightarrow R^f = \frac{1}{E_t [m_{t+1}]}$$
(2.11)

If a risk-free security is not traded, then R^f is simply the "shadow" risk-free rate. Using the definition of covariance $cov(m_{t+1}, x_{t+1}) = E_t[m_{t+1}x_{t+1}] - E_t[m_{t+1}]E_t[x_{t+1}]$, the pricing equation (2.6) is transformed:

$$p_t = \frac{E_t[x_{t+1}]}{R^f} + cov(m_{t+1}, x_{t+1})$$
(2.12)

The price of an asset consists of two terms. The first term expresses the present value of the expected payoff discounted with the risk-free rate - this would be the price of the asset in a risk-neutral world. The second term is an adjustment for the risk. Without losing generality, we substitute m_{t+1} in terms of consumption, as to better interpret the risk adjustment:

$$p_t = \frac{E_t[x_{t+1}]}{R^f} + \frac{cov(\beta u'(c_{t+1}), x_{t+1})}{u'(c_t)}$$
(2.13)

When the covariance is negative, consumption varies positively with payoff, meaning that the asset pays off well when the consumption is already high; and pays off badly when the consumption is already low. This makes the consumption stream more volatile, and since investors do not like uncertainty about consumption, the asset's price must be lowered for the investors to hold it. On the other hand the price of the asset is raised when the covariance is positive: the asset pays offs well when consumption is low, and pays off badly when consumption is already high. Such assets smooth consumption and are more valuable than the expected payoff might indicate.

From the analysis above, it should be obvious that under the consumption-based asset pricing model, it is the covariance of the payoff with the discount factor that determines an asset's riskiness and price - rather than its volatility. Decision makers do not care about intermediate goals such as the volatility of individual assets or portfolios, as long as they manage to keep a steady consumption. Furthermore, only the component of the payoff that is perfectly correlated with the discount factor generates an extra return. Idiosyncratic risk, uncorrelated with the discount factor generates no premium (Cochrane, 2005). Applying the covariance decomposition into the return equation (2.9) we obtain:

$$1 = E_t(m_{t+1})E_t(R_{t+1}^i) + cov(m_{t+1}, R_{t+1}^i)$$
(2.14)

where R_{t+1}^i is the return of an individual asset. The superscript *i* is used to denote this. Rearranging the above equation and using $R^f = \frac{1}{E_t[m_{t+1}]}$ we obtain:

$$E_t(R_{t+1}^i) - R^f = -\frac{cov(m_{t+1}, R_{t+1}^i)}{E_t[m_{t+1}]}$$
(2.15)

The above equation leads to exactly the same conclusions as the basic price equation (2.13). Assets whose returns co-vary negatively with the stochastic discount factor make consumption more volatile, therefore must promise an excess return over the risk free rate to induce investors to hold them. On the other hand, assets that co-vary positively with the discount factor, smooth consumption stream, and offer expected rate of return than can be lower than the risk-free rate or even negative!

Using the definition of covariance $cov(m_{t+1}, R_{t+1}^i) = \rho_{m,R^i}\sigma(m_{t+1})\sigma(R_{t+1}^i)$, we obtain:

$$E_t(R_{t+1}^i) - R^f = -\rho_{m,R^i} \frac{\sigma(m_{t+1})}{E_t[m_{t+1}]} \sigma(R_{t+1}^i) (2.16)$$

Slope $\sigma(m_{t+1})/E(m_{t+1})$ is commonly referred to as the market price of risk. All assets have an expected return equal to the risk-free rate plus a risk adjustment, which is simply the product of the market price of risk and the amount of risk the asset bears. Term $-\rho_{m,R^i}\sigma(R_{t+1}^i)$ is interpreted as the quantity of risk in each asset. Since $-1 \leq \rho_{m,R^i} \leq 1$ the maximum available risk adjustment is limited:

$$\left| E_t \left(R_{t+1}^i \right) - R^f \right| \le \frac{\sigma(m_{t+1})}{E_t[m_{t+1}]} \sigma(R_{t+1}^i)$$
(2.17)

Figure 2.1 is the graphical representation of equation (2.17). All assets' returns must lie within the wedged section that is defined by the slope $\sigma(m_{t+1})/E(m_{t+1})$. Assets whose

return is perfectly correlated with the discount factor have the maximum available risk correction and lie on the mean-variance frontier.



Figure 2.1: The Mean-Variance Frontier

The ratio of an asset's the excess return to its standard deviation is known as the Sharpe Ratio and is limited by the market price of risk:

$$\frac{\sigma(m_{t+1})}{E_t[m_{t+1}]} \ge \frac{\left|E_t(R_{t+1}^i) - R^f\right|}{\sigma(R_{t+1}^i)}$$
(2.18)

The above formula is also known as the Hansen-Jagannathan bounds and can be read twice fold. Given a set of returns, the above equation poses a limit on the set of discount factors that can price them. Again, given a set of discount factors, there is a limit on set of returns that can emerge. Using the risk free rate definition, we obtain:

$$\sigma(m_{t+1})R^{f} \ge \frac{\left|E_{t}\left(R_{t+1}^{i}\right) - R^{f}\right|}{\sigma(R_{t+1}^{i})}$$
(2.19)

Comparing with historical data from the U.S. economy, stock returns averaged about 9% with a standard deviation of about 16% during the past 50 years. At the same time, real annual return on treasury bills was 1%. Applying these numbers on the above equation we obtain a market price of risk 0.5, implying a highly volatile discount factor, too high to be reconciled with any model that connects the discount factor with macroeconomic variables.

This also implies that a representative consumer with power utility function would have a risk aversion parameter γ of 50! This is also known as the *equity premium puzzle*.

2.2 Robust Permanent Income and Pricing

In the work "Robust Permanent Income and Pricing" (1999), Hansen, et al. (henceforth HST) adjust the stochastic discount factor as to include a component accounting for the decision maker' preference for "robustness". Decision makers evaluate the value of an asset based on their expectation about marginal utility growth and the asset's future payoff, conditional on the information set available at that moment. Although not explicitly stated, behind this behavior lies a "model", a specification of how to transform the information available into the expected value. Agents treat their model as a good approximation to an unknown "true" model; however, they often doubt it, fearing that data will come from an unknown model close to the approximating one. Fears about model uncertainty make agents want decision rules that work well for a set of models close to their approximating model. This precautionary behavior is incorporated into the stochastic discount factor. According to HST's proposal, the stochastic discount factor consists of two multiplicative components, the "ordinary" intertemporal rate of substitution adjusting for risk as discussed in the previous section, and the robustness premium that reflects agent's aversion to model uncertainty. Under this approach, the market price of risk is brought much closer to empirical observations, helping explain the equity premium puzzle.

HST laboratory consists of a permanent income model of the economy, originating from the work of Hall (1978). According to the permanent income hypothesis, consumers form expectations about their ability to consume in the future, and based on that, adjust their current consumption. The decision rules are formed as a planning problem – consumers use a model of the economy to derive expectations about the future and then solve a recursive linear quadratic problem. The state transition equation is:

$$x_{t+1} = Ax_t + Bi_t + Cw_{t+1} \tag{2.20}$$

where x_t is the state vector, i_t is the control vector, and w_{t+1} is an i.i.d. Gaussian random variable with $Ew_{t+1} = 0$ and $Ew_{t+1}w'_{t+1} = I$. The decision maker has full visibility of the state vector, which includes variables of consumption with habit persistence, capital stock

and preference shocks, as derived under the permanent income model. Under a *rational expectations* perspective, preferences are modeled as a *risk sensitive* control problem: the agent fully trusts his model and maximizes his intertemporal utility index:

$$U_t = u(i_t, x_t) + \beta R_t(U_{t+1})$$
(2.21)

where:

$$R_t(U_{t+1}) \equiv \frac{2}{\sigma} \log E_t \left[e^{\frac{\sigma U_{t+1}}{2}} \right] (2.22)$$

and E_t is conditional on the state equation (2.20). Utility function (2.21) denotes *risk sensitive* preferences, and is fully discussed in the work of Epstein and Zin (1989), Weil (1993) and Hansen and Sargent (1995). *Risk sensitivity* parameter σ is introduced as to impose an additional risk adjustment to future states, over and above the on that induced by the shape of the standard utility function $u(\cdot)$. Values of σ <0 correspond to a risk averse behavior, while σ =0 leads to the usual von Neumann-Morgenstern form of state additivity. The recursive formulation with a penalty on future states cancels the indifference in timing and implies a preference for early benefits.

The decision maker is maximizing the utility index U_t by choosing a control process i_t adapted to the state transition equation (2.20). Let U_t^e denote the optimal utility index. Hansen and Sargent (1995) provide formulas for Ω and ρ such that:

$$U_t^e = x_t' \Omega x_t + \rho \tag{2.23}$$

Glover and Doyle (1988) linked the stochastic risk-sensitive control problem to robust control. Inspired by this, HST consider the robust control problem. A decision maker with ordinary preferences fears that his model as expressed by state transition dynamics (2.20) has specification errors. He seeks a decision rule for i_t that does better when the miss-specifications materialize, willing to sacrifice optimality when these are absent.

Following the robust control theory, robustness is introduced through a zero-sum two player game. The first player is the decision maker who is trying to maximize his regular utility function while the second player is a malevolent agent, who through his choice of shocks minimizes the utility function. Positing a malevolent agent is just a device that the decision maker uses to perform a systematic analysis of the fragility of his decision rules and to construct a lower bound on the performance it can be attained by using them. The transitory dynamics are defined as:

$$x_{t+1} = Ax_t + Bi_t + C(w_{t+1} + u_t)$$
(2.24)

where u_t is the potential model misspecification, and is also the malevolent agent's control. The decision maker maximizes a "regular" utility function with no risk-adjustment:

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t u(i_t, x_t)$$
 (2.25)

The expectation is conditional on (2.24). The set of misspecifications the decision maker wants to guard against can be defined by imposing a constraint on the size of u_t :

$$E_t \sum_{j=0}^{\infty} \beta^j u_{t+j} \cdot u_{t+j} \le \eta_t \tag{2.26}$$

$$u_t \cdot u_t = \eta_t - \beta \eta_{t+1} \tag{2.27}$$

The above specification allows u_t to feedback on x_t , including its endogenous components. Note that no structure is imposed on u_t , just a constrain on its size, allowing for a wide class of misspecifications. Decision maker's problem is then:

$$\max_{i_t} \min_{u_t} U_0 (2.28)$$

s.t. (2.25) - (2.27)

Hansen and Sargent (1998) formulate this problem recursively and link the risk sensitive and robust formulations by letting $-1/\sigma$ be the Lagrange multiplier of the constraint (2.26) to the optimization problem. They prove that the optimal utility index is:

$$\widehat{U}_t^e = x_t' \Omega x_t + \widehat{\rho} \tag{2.29}$$

Optimal utility index under risk sensitivity U_t^e and under robust preferences \hat{U}_t^e share the same matrix Ω but have different constants ρ and $\hat{\rho}$. This relationship implies that the risk-sensitive preferences are exactly the same as the robust control preferences, leading to the

same i_t , and bearing the same implications for consumption and savings. The solution for \hat{u}_t , the optimal control of the malevolent agent is given by:

$$\hat{u}_t = \sigma (I - \sigma C' \Omega C)^{-1} C' \Omega (A - BF) x_t$$
(2.30)

where $i_t = -Fx_t$ denotes the optimal decision rule, common in both formulations. Note that malevolent agent's control \hat{u}_t is linked to the risk sensitive problem through the risk sensitivity parameter σ .

Based on this equivalence, HST formulate a risk-sensitive version of the permanent income model with habit persistence and estimate it using time series data for consumption and investment from the US economy. In order to obtain asset prices, the optimal resource allocation problem is considered, and the solution is obtained from the robust permanent income problem. A large number of identical agents trade in competitive security markets where quantities are equilibrium allocations for a competitive equilibrium. Since agents are identical, equilibrium prices become "shadow prices" that clear the market. HST demonstrate that the risk-sensitivity parameter σ and the subjective discount factor β are not separately identifiable. Pairs of (σ, β) are observationally equivalent for quantities, however they bear different implications on the pricing of risky assets and on the amount of model misspecification that is required to justify the equivalence to risk sensitivity. For every variation in σ there is an offsetting change in β that leaves decision rule *F* and consumption, investment and all quantities unaltered.

HST consider the consumption based model to price assets with an multiplicative adjustment to the stochastic discount factor that reflects agents' aversion to model uncertainty. Specifically, they define:

$$m_{t+1} = m_{t+1}^J m_{t+1}^u \tag{2.31}$$

Where m_{t+1}^u is the Radon-Nikodym derivative (or likelihood ratio) of the distorted conditional probability of x_{t+1} with respect to the approximating conditional probability:

$$m_{t+1}^{u} = \frac{e^{\left[-\frac{1}{2}(w_{t+1}-\hat{u}_{t})'(w_{t+1}-\hat{u}_{t})\right]}}{e^{\left[-\frac{1}{2}w_{t+1}'w_{t+1}\right]}}(2.32)$$

It simply denotes the density ratio of the "distorted" probability distribution relative to the "true" one. Note that the likelihood ration m_{t+1}^u depends on the risk sensitivity parameter through equation (2.30). m_{t+1}^f is the familiar intertemporal rate of substitution:

$$m_{t+1}^f = \beta \frac{M_{t+1}^c}{M_t^c} (2.33)$$

where M_{t+1}^c is the marginal utility of consumption. The basic pricing equation becomes:

$$p_t = E_t \Big[m_{t+1}^f m_{t+1}^u x_{t+1} \Big]$$
(2.34)

while the expectation is conditional on the robust problem formulation. Recall the Hansen-Jagannathan bounds:

$$\frac{\sigma(m_{t+1})}{E_t[m_{t+1}]} \ge \frac{\left|E_t(R_{t+1}^i) - R^f\right|}{\sigma(R_{t+1}^i)}$$
(2.35)

By a straightforward calculation of (2.32), it follows that:

$$E_t[(m_{t+1}^u)^2] = e^{\hat{u}_t'\hat{u}_t}$$
(2.36)

Because $E_t(m_{t+1}^u) = 1$ by construction, it follows that the conditional standard deviation of m_{t+1}^u is:

$$\sigma_t(m_{t+1}^u) = \sqrt{e^{\hat{u}_t'\hat{u}_t} - 1} \approx \left|\hat{u}_t\right|$$
(2.37)

HST call $\sigma_t(m_{t+1}^u)$ the market price of uncertainty, and is measured by the size of distortion the representative investor considers. The fact that a preference for guarding against the worst case model could lead to a direct enhancement of the market price of risk is not surprising. Concern for robustness directs the associated pessimism to the mean-standard deviation frontier, amplifying what is usually interpreted as the market price of risk.

Recall that empirical data applied on the Hansen-Jagannathanbounds implied a highly volatile discount factor that under the standard consumption based model lead to a risk aversion parametery of 50. The above formulation helps explain the equity premium puzzle

by compensating for the low theoretical value of the "ordinary" discount factor m_{t+1}^{f} . Given the "pessimistic" construction of u_t , someone might expect the two components m_{t+1}^{f} and m_{t+1}^{u} to be positively correlated, increasing the total effect on the discount factor.

In their later work, Barillas, Hansen and Sargent (2007) go as far as to suggest that the market price of risk is largely compensating the representative consumer for bearing model uncertainty and not risk. Modest amount of model uncertainty can substitute for large amounts of risk aversion in terms of its effects on asset prices. Asset prices cannot provide useful information about consumer's attitude toward risk.

Chapter 3: Robust Portfolio Rules and Asset Pricing

In this chapter, the recent work of P. J. Maenhout "Robust Portfolio Rules and Asset Pricing" (2004) is summarized and discussed. The focus of this paper is asset pricing and portfolio and consumption rules when the decision maker worries about the return process of the risky asset. There are many good reasons for the investors to doubt the expected equity return and want to take decisions that work well under this kind of uncertainty. As a first step, Maenhout considers and extends the framework of Anderson et al. (2003) (henceforth AHS) on robust preferences and introduces homothetic robust preferences. An analytical solution in closed form indicates that robustness decreases the ratio of wealth invested in the risky asset. This kind of behavior and overall results bear close resemblance to Stochastic Differential Utility (henceforth SDU) - the continuous-time version of recursive utility, separating intertemporal substitution and risk aversion - as introduced by Epstein and Zin (1989) and Weil (1990), with different implications however on the risk aversion coefficient required to reconcile the excess return premium. As Maenhout wants to distinguish these three distinct behaviors, risk aversion, intertemporal substitution and uncertainty aversion, he considers a decision maker with SDU and robust preferences. Robustness can be interpreted as increasing the "effective risk aversion" as to include the uncertainty aversion, without changing the will to substitute intertemporally. An endowment economy is also considered in the Style of Lucas (1978) as to derive asset prices in equilibrium. His main conclusion is that both market risk and model uncertainty are priced in equilibrium. Robustness also drives down the risk free rate through precautionary savings. By trying to measure "risk aversion" from empirical data, one would obtain the combined effect of risk aversion γ and uncertainty aversion θ or simply $\gamma + \theta$. Stylized experiments that try to estimate "risk aversion" of individuals, usually report relative small values of γ because they involve situations where events are well specified and usually this environment-specific preference for robustness is not exhibited. Finally, empirical data from the US economy are used to calibrate the equilibrium model. Given the observed risk free rate and excess return of the risky asset from two different time periods, the preference parameters (time, risk, uncertainty and intertemporal substitution) are adjusted as to satisfy the equilibrium model. Robustness helps resolve both the risk-free rate and the equity premium puzzle, as relatively reasonable parameter values reconcile the observational data.

Until recently, the bulk of work on dynamic portfolio choice assumes that point estimates for the asset return parameters are known and fixed. However, first moments are notoriously hard to estimate (Merton (1980), Cochrane (1998), Blanchard (1993)) while some authors express skepticism about the reliability of historical estimates (Heaton and Lucas (1999) and Campbell and Shiller (1998)). It is only natural to seek decision rules that work reasonably well if there is some misspecification about the return process.

Consider an agent who faces a simple dilemma: how much to save and how much to consume. He is having at his disposal two financial assets - a riskless paying a risk-free rate r_f and a risky one paying an excess return $\mu - r_f$. His objective is to maximize the expected life time utility over consumption:

$$E_t \left[\int_0^T e^{-\delta t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right], \tag{3.1}$$

Preferences are modeled with the help of the power utility function, with γ being the relative risk aversion co-efficient, C_t the consumption at time t and $\delta > 0$ the discount factor indicating time preference. Expectation is conditional on information at time t. The price of the risky asset evolves according to the standard Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{3.2}$$

The agent simply decides how much to consume and how much to invest in the risky asset. The state equation for wealth is:

$$dW_t = \left[W_t \left(r + a_t (\mu - r)\right) - C_t\right] dt + a_t \sigma W_t dB_t (3.3)$$

where a_t denotes the fraction of wealth invested in the risky asset at time t. Merton (1971) considered a value function V(W, t) to pose the above multi-period problem as a two period problem linking it to optimal control theory and showing that the solution can be obtained by solving the Hamilton-Jacobi-Bellman (henceforth HJB) equation for optimality:

$$0 = \sup_{a_t, c_t} \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \delta V(W, t) + D^{(a, C)} V(W, t) \right]$$
(3.4)

with boundary conditions V(W,T) = 0 at terminal time T. $D^{(a,C)}V(W,t)$ can be thought as the differential expectations operator $\frac{1}{dt}E(dV)$ and is provided by:

$$D^{(a,C)}V(W,t) = \frac{\partial V}{\partial W} \Big[W_t \left(r + a_t (\mu - r_f) \right) - C_t \Big] + V_W + \frac{1}{2} V_{WW} a_t^2 \sigma^2 W_t^2 \qquad (3.5)$$

where V_W is the partial derivative of the value function with respect to wealth, i.e. $\frac{\partial V}{\partial W}$. The HJB equation incorporates the assumption that wealth evolves according to (3.3), reflecting a particular underlying model. However, for the reasons discussed before, the decision maker wants to take uncertainty about the return process into account when formulating portfolio rules. He suspects that the model is misspecified and considers alternative models that are relative close to the reference model. Following robust control literature, this preference for robustness is modeled through a malevolent agent who through his control is minimizing the expected life-time utility of the decision maker. The malevolent agent's control is constrained by an entropy penalty denoting the strength of the preference for robustness.

Inspired by AHS, the transition law of wealth is modified as to include an endogenous drift to the Brownian motion:

$$dW_t = \left[W_t \left(r + a_t (\mu - r)\right) - C_t\right] dt + a_t^2 \sigma^2 W_t^2 u(W_t) dt + a_t \sigma W_t dB_t (3.6)$$

where $u(W_t)$ is the control of the malevolent agent, through which the preference for robustness is introduced. The HJB equation then becomes:

$$0 = \sup_{a_t, c_t} \inf_{u(W_t)} \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \delta V(W, t) + D^{(a,c)} V(W, t) + V_W a_t^2 \sigma^2 W_t^2 u(W_t) + \frac{1}{2\Psi(W, t)} a_t^2 \sigma^2 W_t^2 u^2(W_t) \right] (3.7)$$

The first three terms are exactly the same as in the original HJB equation, while the fourth is the adjustment to the differential operator (3.5) reflecting the additional drift to the transitional law of wealth as in (3.6). The last term $\frac{1}{2\Psi(W,t)}a_t^2\sigma^2W_t^2u^2(W_t) > 0$ is called *entropy penalty* and is used to constrain the disparity between the reference model and the

worst case alternative model that is considered, by imposing a penalty on alternative models that are *too far away* from the reference model. The distance of two models is measured by the derivative of relative entropy and in the above formulation is introduced by term $a_t^2 \sigma^2 W_t^2 u^2(W_t)$. Intuitively it can be though as a log-likelihood ratio. The parameter $\Psi(W,t)>0$ measures the strength of the preference for robustness - with higher values corresponding to a stronger preference. The entropy penalty - the distance between models weighted by the preference for robustness - is always positive or zero; large drift distortions work in the opposite direction of the malevolent agent's strategy for expected utility minimization. A stronger preference for robustness ($\Psi(W,t)>>0$), would imply a reduced impact of the entropy penalty, allowing the decision maker to consider and guard against larger drift distortions (alternative models with larger distance). Finally, note that $\Psi(W,t)=0$ corresponds to expected utility maximization. Solving for the minimization part of the HJB equation yields:

$$u^{*}(W_{t}) = -\Psi(W, t)V_{W}$$
(3.8)

In case the decision maker has no preference for robustness ($\Psi(W, t) = 0$), then there are no perturbations to guard against ($u^*(W_t) = 0$). Substituting (3.8) into the HJB we obtain:

$$0 = \sup_{a_t, C_t} \left[\frac{C_t^{1-\gamma}}{1-\gamma} - \delta V(W, t) + D^{(a, C)} V(W, t) - \frac{\Psi}{2} V_W^2 a_t^2 \sigma^2 W_t^2 \right] (3.9)$$

with boundary conditions V(W,T) = 0 at terminal time *T*. The necessary optimality conditions for consumption and portfolio choice are:

$$(C_t^*)^{-\gamma} = V_W \tag{3.10}$$

$$a_t^* = \frac{-V_W}{[V_{WW} - \Psi V_W^2]W} \frac{\mu - r}{\sigma^2}$$
(3.11)

Under the original Merton's formulation (with no preference for robustness), the optimality condition for the fraction of wealth invested in the risky asset is:

$$a_t^* = \frac{-V_W}{V_{WW}W} \frac{\mu - r}{\sigma^2} \tag{3.12}$$

where $\frac{\mu - r}{\sigma^2}$ is the mean variance frontier and $\frac{-V_{WW}W}{V_W}$ adjusts for risk aversion. The preference for robustness simply introduces the additional term $\Psi V_W W > 0$ to the "risk aversion" adjustment. On the other hand, the optimality condition for consumption is not affected by the introduction of robustness. By substituting the optimality conditions in the HJB equation (3.9), it becomes clear that no analytical solution exists for the value function, unless further assumptions are made. Without knowledge of the value function (or at least its partial derivatives), the impact of introducing robustness on portfolio choice cannot be quantified. AHS consider a fixed value $\theta > 0$ for parameter Ψ , independent of wealth and constant across time. However, Maenhout proves that preferences induced in that case are not homothetic, since the portfolio weight is not independent of wealth (at least not for general CRRA preferences) and robustness wears off as wealth rises. Homothetic preferences are desired not only out of the modeling convenience, but for a number of reasons; they imply stationary rate of returns, wealth invariance of optimal decisions and finally facilitate aggregation and the construction of a representative agent. In order to impose the desired homotheticity, Maenhout proposes the following modification, scaling θ by the value function:

$$\Psi(W,t) = \frac{\theta}{(1-\gamma)V(W,t)} > 0 \tag{3.13}$$

The HJB equation (3.9) then becomes:

$$0 = \sup_{a_t, C_t} \left[U(C_t) - \delta V(W, t) + D^{(a, C)} V(W, t) - \frac{\theta}{2(1 - \gamma)V(W, t)} V_W^2 a_t^2 \sigma^2 W_t^2 \right] (3.14)$$

Solving the above equation for $U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$, one obtains:

$$V(W,t) = \left[\frac{(1 - e^{-b(T-t)})}{b}\right]^{\gamma} \frac{W^{1-\gamma}}{1-\gamma}$$
(3.15)

where

$$b \equiv \frac{1}{\gamma} \left[\delta - (1 - \gamma)r - \frac{1 - \gamma}{2[\gamma + \theta]} \left[\frac{\mu - r}{\sigma} \right]^2 \right] (3.16)$$

The optimal consumption rule has the same structure as the one obtained from the original problem posed by Merton, reflecting however a different portfolio weight:

$$C_t^* = \frac{b}{1 - e^{-b(T-t)}} W_t \tag{3.17}$$

The optimal portfolio rule is provided by:

$$a^* = \frac{1}{\gamma + \theta} \frac{\mu - r}{\sigma^2} \tag{3.18}$$

Again, the above has exactly the same form as the portfolio rule derived from Merton solution. However, the risk aversion adjustment term $\frac{1}{\gamma+\theta}$ also reflects the preference for robustness. Since $\gamma + \theta > \gamma$ the proportion of wealth invested in the risky asset is smaller than the one provided from the Merton solution.

Financial markets are environments where decision makers are faced with uncertainty. Probabilities may not always be known or can be estimated, while when there is data available, decision makers often doubt them. By trying to measure "risk aversion" from empirical data, one would obtain the combined effect of risk aversion γ and uncertainty aversion θ or simply $\gamma + \theta$. Although stylized experiments have been designed to estimate "risk aversion" of individuals, they usually involve situations where events are well specified and usually this preference for robustness is not exhibited. Therefore, the "risk aversion" parameter captured from such experiments is simply γ . The combined effect of risk and uncertainty aversion present in financial markets cannot be measured through such experiments. Uncertainty aversion is environment specific, and although risk aversion γ might be constant across environments, difficult situations might generate a higher perceived $\gamma + \theta$.

Although not explicitly specified until now, a possible range for "empirical" values of the parameter of risk aversion θ under this model can be obtained by solving for the worst case expected excess return of the risky asset the investor worries about. Consider the state equation for wealth where the investor optimally allocates his wealth between the riskless asset and the risky assetby choosing a^* under the worst case scenario (malevolent agent employs his optimal control u^*):

$$dW_t = \left[W_t \left(r + a_t^*(\mu - r)\right) - C_t\right] dt + (a_t^*)^2 \sigma^2 W_t^2 u^*(W_t) dt + a_t^* \sigma W_t dB_t \quad (3.19)$$

The above can be interpreted as if the decision maker would expect the price of the risky asset to evolve (assuming no dividends) according to:

$$\frac{dS_t}{S_t} = \left[\mu + \alpha^* W \sigma^2 u^*\right] dt + \sigma dB_t \tag{3.20}$$

And by substituting the values of α^* and u^* :

$$\frac{dS_t}{S_t} = \left[\mu - (\mu - r)\frac{\gamma}{\gamma + \theta}\right]dt + \sigma dB_t$$
(3.21)

The above equation implies that the decision maker worries that the return premium of the risky asset in not $(\mu - r)$ and is guarding against a worst case expected excess return equal to:

$$EP_{P} \equiv E_{t}^{u^{*}} \left[\frac{dS_{t}}{S_{t}} - rdt \right] = \frac{\gamma}{\gamma + \theta} (\mu - r)dt$$
(3.22)

where $E_t^{u^*}$ is the conditional expectation under the worst case alternative model the decision maker considers. By re-arranging the above equation, a range for the possible values of θ can be obtained:

$$\theta = \gamma \frac{EP_T - EP_P}{EP_P} \tag{3.23}$$

where $EP_T = (\mu - r)$ is the equity premium in the absence of uncertainty. To better understand the above results, consider the case where the investor is moderately risk averse with γ =5. By using historical data from the US economy, the stock returns averaged about 9% with a standard deviation of about 16% during the past 50 years. At the same time, real annual return on treasury bills was 1%, meaning that the observed equity premium is 8%. Table 1 summarizes implications for portfolio rules and expected equity premium when the decision maker is uncertainty averse.

	proportion of wealth	expected excess return	
Uncertainty aversion	invested in risky asset $lpha^*$	of risky asset (EP_P)	
θ=0	62.5%	8.0%	
θ=0.1	61.3%	7.8%	
θ=0.5	56.8%	7.3%	
θ=1	52.1%	6.7%	
θ=2.5	41.7%	5.3%	
θ=5	31.3%	4.0%	

<u>Table 1</u>: Proportion of wealth invested in risky asset and expected excess return under homothetic robustness when risk aversion parameter γ =5 and observed excess return ($\mu - r$) = 8%.

The first line of the above table - where the parameter of uncertainty aversion θ =0 - simply denotes the expected utility case without any preference for robustness. The decision maker trusts the data available, expecting an excess return of the risky asset equal to (µ-r), and invests a significant proportion of his wealth to the risky asset. However, in the cases where the investor is more uncertainty averse, he considers perturbations in the model, leading to a smaller expected excess return of the risky asset. He guards against it by investing a smaller proportion of his wealth to the risky asset. The more uncertainty averse the decision maker is, the less proportion of his wealth is willing to invest in the risky asset.

As a next step, Maenhout demonstrates that the investor with a homothetic preference for robustness is closely related to an investor with stochastic differential utility form (SDU) as described by Duffie and Epstein (1992a). As also noted by AHS, the HJB equation (3.14) describing the preferences of the uncertainty averse agent has exactly the same form as a Duffie-Epstein agent with elasticity of intertemporal substitution γ^{-1} and coefficient of relative risk aversion $\gamma + \theta$, leading to the same consumption rule. However, there is a crucial difference between SDU and a preference for robustness. Stochastic Differential Utility reconciles empirical observations by simply predicting a very high value of γ , which is in contrast to low estimates based on stylized experiments. Furthermore, in the non-robust stochastic differential utility framework, a power utility function implies equal willingness for substitution over time as across states, since the coefficient of relative risk aversion γ is also the inverse of elasticity of intertemporal substitution. On the other hand, robust preferences

generate environment specific "risk (uncertainty) aversion", increasing the total "effective risk aversion($\gamma + \theta$)", predicting an agent less willing to substitute across states - as the coefficient of relative risk aversion becomes $\gamma + \theta > \gamma$ -without changing the willingness to substitute intertemporally. Having intertemporal substitution decoupled from risk aversion will prove important later on.

In order to distinguish between these three distinct behaviors – risk aversion, intertemporal substitution and uncertainty aversion – Maenhout extends the SDU framework by considering a Duffie-Epstein agent who also worries about model uncertainty. This can be done by replacing $U(C_t) - \delta V(W, t)$ using the normalized Duffie Epstein aggregator in the HJB equation (3.14):

$$0 = \sup_{a_t, C_t} \left[\frac{1}{1 - \psi} \left\{ \frac{c^{1 - \psi}}{\left((1 - \gamma) V(W, t) \right)^{\frac{\gamma - \psi}{1 - \gamma}}} - \delta(1 - \gamma) V(W, t) \right\} + D^{(a, C)} V(W, t) - \frac{\theta}{2(1 - \gamma) V(W, t)} V_W^2 a_t^2 \sigma^2 W_t^2 \right] (3.24)$$

Where θ is the robustness parameter, γ is the risk aversion parameter and ψ^{-1} denotes the elasticity of intertemporal substitution. The optimal portfolio and consumption rules are given by:

$$C_t^* = \frac{d}{1 - e^{-d(T-t)}} W_t \tag{3.25}$$

$$a^* = \frac{1}{\gamma + \theta} \frac{\mu - r}{\sigma^2} \tag{3.26}$$

where $d \equiv \frac{1}{\psi} \left[\delta - (1 - \psi)r - \frac{1 - \psi}{2[\gamma + \theta]} \left[\frac{\mu - r}{\sigma} \right]^2 \right]$. The above decision rules allow Maenhout to formalize the link between stochastic differential utility and robust preferences: An investor with a homothetic preference for robustness $\Psi(W, t) = \frac{\theta}{(1 - \gamma)V(W, t)}$ and Duffie-Epstein utility function with risk aversion γ and elasticity of intertemporal substitution ψ^{-1} is observationally equivalent to a Duffie-Epstein investor with the same elasticity of intertemporal substitution ψ^{-1} and coefficient of relative risk aversion $\gamma + \theta$.

In order to derive the equity premium and the risk free rate in equilibrium, an endowment economy as described by Lucas (1978) is then considered. The representative agent receives an endowment, which he has to fully consume in equilibrium, and can trade a riskless-free asset and a risky one, entitling the owner to a dividend (the endowment). Both risk free rate and equity premium adjust as to support a no-trade equilibrium. The dividend is modeled by a geometric Brownian motion:

$$dD_t = \mu_D D_t dt + \sigma_D D_t dB_t \tag{3.27}$$

with μ_D , $\sigma_D > 0$. The total return of the risky asset, consisting of both the dividend yield and the capital gain evolves again according to a Brownian motion:

$$\frac{dS_t + D_t dt}{S_t} = \mu_s dt + \sigma_s dB_t \tag{3.28}$$

where coefficients μ_s and $\sigma_s > 0$ are obtained by equilibrium condition. Denoting the riskfree rate by r_f and a_t the fraction of wealth allocated to the risky asset, the representative agent's wealth evolves according to:

$$dW_t = \left[W_t\left(r_f + a_t(\mu_s - r)\right) - C_t\right]dt + a_t\sigma_s W_t dB_t$$
(3.29)

The HJB equation then for a robust investor with Duffie-Epstein utility, intertemporal substitution of risk aversion ψ^{-1} , risk aversion γ , and preference for robustness θ is:

$$0 = \sup_{a_t, C_t} \left[\frac{1}{1 - \psi} \left\{ \frac{c^{1 - \psi}}{\left((1 - \gamma) V(W, t) \right)^{\frac{\gamma - \psi}{1 - \gamma}}} - \delta(1 - \gamma) V(W, t) \right\} + V_w \left[W_t \left(r_f + a_t(\mu_s - r) \right) - C_t \right] + \frac{1}{2} \left(V_{ww} - \frac{\theta}{(1 - \gamma) V(W, t)} V_W^2 \right) a_t^2 \sigma^2 W_t^2 \right] (3.30)$$

The robust equilibrium is then defined as a set of rules over consumption C_t , investment a_t , prices S_t and risk free rate r_f such that simultaneously:

• Markets clear continuously: $C^* = D$ and $a^* = 1$

• The HJB equation (3.30) is solved subject to boundary condition $\lim_{t\to\infty} E[e^{-\delta t}V(W_t)] = 0$

The optimality conditions are:

$$a^* = \frac{1}{\gamma + \theta} \frac{\mu_s - r_f}{\sigma_s^2} \tag{3.31}$$

$$C_t^* = d_s W_t \tag{3.32}$$

where $d_s \equiv \frac{1}{\psi} \left[\delta - (1 - \psi)r - \frac{1 - \psi}{2[\gamma + \theta]} \left[\frac{\mu_s - r_f}{\sigma_s} \right]^2 \right]$. From the clearing of market $(a^* = 1)$ and (3.31), the CCAPM result is easily obtained:

$$\mu_s - r_f = [\gamma + \theta]\sigma_s^2 \tag{3.33}$$

It follows that the excess return on the risky asset evolves according to:

$$\frac{dS_t + D_t}{S_t} - r_f dt = [\gamma + \theta]\sigma_{CS}dt + \sigma_D dB_t$$
(3.34)

where $\sigma_{CS} \equiv cov\left(\frac{dc}{c}, \frac{ds}{s}\right)$. Both market risk and model uncertainty are priced in equilibrium with the combined price of risk given by $\gamma + \theta$, higher than what would be expected by genuine risk aversion alone. The equilibrium risk free rate is given by:

$$r_f = \delta + \psi \mu_D - \frac{1}{2} [1 + \psi] [\gamma + \theta] \sigma_D^2(3.35)$$

The risk-free rate is determined by three fundamental factors of savings in the economy:

• Time preference δ . When people are impatient (high δ) it takes a high real interest rate to convince them to save.

• Intertemporal substitution and consumption growth. When people expect consumption growth in the future, it takes a higher risk free rate to convince them to save.

• Volatility of consumption. People do not like uncertainty about consumption and want to save more, driving down the equilibrium risk free rate through precautionary savings.

Decoupling intertemporal substitution from risk aversion is very important since it allows high values of γ without implicitly producing a high risk free rate. This will become particular useful in the empirical calibration later on. In order to derive the *worst case* expected equity premium in under equilibrium, Maenhout considers the least favorable excess return. This is obtained whenall the agents in the model do not exhibit a preference for uncertainty aversion, i.e. $\theta = 0$. In that case, the *worst case* expected equity premium is given by:

$$EP_P^* = \gamma \sigma_{cs}(3.36)$$

Although the endogenous equity premium depends on uncertainty aversion, the pessimistic equity premium in the economy equilibrium does not depend on θ . What θ does is to index the distance between the pessimistic equity premium and the true equity premium.

The equilibrium model is then calibrated using empirical data from the US economy. Given the observed risk free rate and excess return of the risky asset from two different time periods, the preference parameters δ , γ , ψ , θ are adjusted as to satisfy the equilibrium equations (3.34) and (3.35). Consumption growth μ_D , volatility σ_D^2 and covariance σ_{cs} are also calculated from empirical data. Discount rate δ must be strictly positive while rate of intertemporal substitution ψ^{-1} is constrained to less than one, following the recent research of Weber (1995) and Vissing-Jorgensen (2002). Only values of γ less than 10 are considered and θ is set to match the historical equity premium. The table below summarizes the preference parameters as derived by Maenhout, based on data taken from Campbell (1999).

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Estimated consumption and return parameters					
Sample:	1891-1994	1947-1996			
μ_c	1.74%	1.91%			
σ _C	3.26%	1.08%			
σ_S	18.53%	15.21%			
ρ_{CS}	0.497	0.193			
r _f	1.96%	0.79%			
$\mu_S - r_f$	6.26%	7.85%			

Table 2: Estimated consumption and return parameters based on two sets of historical data.

Required Preference Parameters						
Sample:	1891-1994		1947-1996			
	Standard	SDU with	Standard	SDU with		
	Expected Utility	robustness	Expected Utility	robustness		
δ	-0.101	0.02	-1.10	0.015		
ψ^{-1}	γ^{-1}	0.6	γ^{-1}	0.6		
γ	21	7	247	10		
θ	0	14	0	237		
EP_P^*	-	2.1%	-	0.32%		

<u>Table 3</u>: Preference Parameters reconciling historical data for standard expected utility model (without a preference for robustness) and SDU with a preference for robustness.

Standard expected utility cannot reconcile either set of historical data since time preference δ turns negative and the risk free rate puzzle shows up. Furthermore, in the case of post war data, the equity puzzle is also present as only a very high risk aversion value would fit the data. Introduction of robustness under the SDU framework (decoupling of risk aversion from intertemporal substitution) allows for plausible values of δ , γ and ψ . The century long sample is easier to match and risk aversion is maintained at the relatively low value of 7. The

"effective price of risk " is $\gamma + \theta = 21$ while values of ψ and δ are also considered plausible, helping explain the low risk free rate despite the precautionary savings imposed by values of $\gamma + \theta$. The worst case expected equity premium is 2.1%, and coincides with the average pessimistic answer in a survey conducted by Welch (2000) among financials economists. On the other hand, in the post war data, the excess return can only be reconciled by a very high "effective price of risk" $\gamma + \theta = 247$. Risk aversion γ takes the maximum value allowed and uncertainty aversion needs to be as high as 237. Despite the large valued of γ , the pessimistic expected equity premium is 0.32% mainly due to the small value of the covariance of consumption and asset prices σ_{CS} . Finally, both δ and ψ take reasonable values.

The final step of calibration is to explore whether the values 14 and 237 of uncertainty aversion θ are plausible. As discussed before, θ is used to index the set of alternative models the agent considers by constraining the distance between the reference model and the worst case alternative model. As a consequence, θ would also dictate the gap between the reference expected equity premium $\mu_s - r_f$ and the and the pessimistic expected equity return $EP_P^* = \gamma \sigma_{cs}$. The decision maker would try to determine whether data are coming from the reference or an alternative model and to this end he can perform likelihood ratios tests based on observations available. The models are difficult to distinguish if the probability of rejecting one in favor of the other is high. Using a Bayesian model selection problem, the difference of 4.16% between the reference equity premium and the pessimistic premium would imply that in the century long data, the decision maker would find it hard to distinguish between the two models so it would be natural to be uncertainty averse. The value of uncertainty aversion $\theta = 14$ is considered plausible. In contrast, in the post war data, the gap between the equity premium of two models is as high as 7.53%. This implies that the decision maker should be able to distinguish between these two models that generate so different results, therefore, the high value of uncertainty aversion $(\theta = 237)$ does not seems plausible.

As a final note, Brown, Goetzman, Ross (1995) and Jorion and Goetzman (1999) argue that the calibration of models using historical data from the US economy, the world's most successful, suffers from severe ex-post survival bias. The *real* observed equity return in the US stock market is 4.3%, far above the median of the world's stock markets of only 0.8% (Jorion and Goetzman (1999)). Investors are uncertain whether the generous equity premium of the US stock market will materialize again, leading to a more cautious behavior. The equilibrium model that is described above formalizes exactly this idea.

Chapter 4: Robust Efficient Frontiers

This chapter briefly presents applications of robust control in mean variance frontier analysis and portfolio selection problems. The first mathematical formulation of allocating capital over a number of available assets with the objective of maximizing the "return" of the investment while minimizing the "risk" is owed to Markowitz (1952, 1959). In the Markowitz portfolio selection problem, the "return" on a portfolio is measured by the expected value of the random portfolio return, and "risk" is quantified by the variance of the portfolio return. The investor is said to have mean-variance preferences if his portfolio preference is based only on the mean return and the volatility of return. A risk averse investor would desire a portfolio with the highest possible return and the lowest possible risk at the same time. However, as there is in general a positive relationship between expected return and risk, both objectives cannot be achieved simultaneously. Given the maximum risk that an investor is willing to tolerate, the optimal allocation can be obtained by solving an optimization problem.

Markowitz also demonstrated that a portfolio of investment assets can have collectively lower risk than any of the individual components, a concept that is known as diversification. Intuitively, this can be explained behind the fact that asset prices move inversely, or at different times, in relation to each other, therefore reducing the total variance of the portfolio return. The strength of this relationship between the assets' price changes is measured by the correlation of the mean returns and defines the benefits of diversification. A key contribution of Markowitz Portfolio Theory is a way to find the best possible diversification strategy so that for a given return, the minimum variance is achieved. This set of all such portfolios is known as the "mean variance efficient frontier; a risk averse rational investor would choose a portfolio from this set based on his risk aversion preferences, trading off between expected return and risk.

Despite the theoretical success of the mean-variance model (Markowitz and Sharpe shared the Nobel Memorial Prize in Economic Sciences for their work on portfolio allocation and asset pricing), practitioners often challenge some of the basics assumptions, doubt its applicability and mistrust the results. One of the key reasons for this is the model's susceptibility to parameter estimation errors. Market parameters, (like the mean return of an asset), are notoriously hard to estimate and subject to statistical errors. However, the optimization problem in the heart of the mean variance efficient frontier is very sensitive to perturbations in the parameters of the model. The following quote of Michaud (1998) is indicative: "Although Markowitz efficiency is a convenient and useful theoretical framework for portfolio optimality, in practice it is an error-prone procedure that often results in *error-maximized* and *investment-irrelevant* portfolios". Various authors has examined the sensitivity of mean variance efficient portfolios to errors in parameters (see for example Klein and Bawa (1976), Best and Grauer (1991), Chopra (1993) and many others). Several techniques have been suggested to reduce the sensitivity of portfolios to *input uncertainty*, with most common the Bayesian approaches (see for example Bawa et al (1979), Pastor (2000), and many others).

Another branch of the parameter sensitivity related literature is inspired from the robust control theory. Robust control theory emerged to explicitly deal with model uncertainty and guarantee a given level of performance as long as the uncertain parameters remain within specific bounds. This chapter briefly discusses the application of robust control in alleviating the sensitivity problem and designing robust mean variance efficient frontiers. The key idea is to incorporate a model of data uncertainty in the formulation of the optimization problem and optimize for the worst case scenario. A recent survey can be found in Fabozzi et al. (2007), while this chapter summarizes Kim and Boyd (2007).

Consider an investor who wants to allocate his savings among N available *risky* assets. Assets' returns are modeled as random variables, with known point estimates for the first and second moments. Consider the mean vector:

$$\mu = \mathbf{E}(\tilde{r}) = \mathbf{E}(\tilde{r}_1 \ \tilde{r}_2 \ \dots \ \tilde{r}_N) = (\mu_1 \ \mu_2 \ \dots \ \mu_N)(4.1)$$

where \tilde{r}_i denotes the random variable describing the return of the i^{th} asset, **E** is the expectation operation, and $\mu_i = \mathbf{E}(\tilde{r}_i)$ the expected return. Covariance matrix is provided by:

$$\Sigma = \mathbf{E}(\tilde{r} - \mu)(\tilde{r} - \mu)^{T} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_{21} & \dots & \sigma_{2N} \\ \dots & \dots & \dots & \dots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_{NN} \end{bmatrix} (4.2)$$

Where σ_{ij} is the covariance of the i^{th} asset with the j^{th} asset and $\sigma_{ii} = \sigma_i^2$ the variance of the i^{th} asset. It is assumed that Σ is positive definite, a property that will prove useful later on in the optimization. For a brief overview of the implications of this property of the correlation matrix in the portfolio variance see Ong and Ranasinghe (2000). Let w_i denote the portion of wealth invested in the i^{th} asset. Obviously $\sum_{i=1}^{N} w_i = 1$ or $\mathbf{1}^T w = 1$, where $\mathbf{1}$ is the vector of all ones. The expected return of the portfolio is then:

$$\mathbf{E}(\tilde{R}_P) = w^T \mu \tag{4.3}$$

The risk of the portfolio is measured by the standard deviation of the return:

$$std(\tilde{R}_P) = (w^T \Sigma w)^{\frac{1}{2}}$$
(4.4)

Given the maximum amount of risk the investor is willing to take (the maximum acceptable volatility level σ^{tol}), the objective of finding portfolio weights that maximize the portfolio's expected return can be posed as an optimization problem:

$$\begin{aligned} \underset{w}{\text{maximize } w^{\text{T}}\mu} & (4.5) \\\\ \text{s. t. } \sqrt{w^{\text{T}}\Sigma w} &\leq \sigma^{\text{tol}} \\\\ 1^{\text{T}}w &= 1 \end{aligned}$$

For every possible level of risk tolerance (value of $\sigma \in \mathbf{R}_+$), the optimal return defines the curve:

$$f_{\mu,\Sigma}(\sigma) = \underset{1^{\mathrm{T}}\mathsf{w}=1, \ \sqrt{\mathsf{w}^{\mathrm{T}}\Sigma\mathsf{w}}\leq\sigma}{\mathrm{maximize}} \mathsf{w}^{\mathrm{T}}\mu(4.6)$$

The concave and increasing part of $f_{\mu,\Sigma}(\sigma)$ is called the *mean variance efficient frontier*, and any portfolio w is called *mean variance efficient* if its risk and mean return are on the efficient frontier. The efficient frontier is the set of all Pareto optimal portfolios; any portfolio with the same return as the mean variance efficient portfolio, would simply have a higher risk. Figure 4.1 show a typical mean variance frontier when there is no risk free asset. The ratio of excess expected return of a portfolio w (relative to the risk free return r_f) to the return volatility is known as the Sharpe Ratio and denotes the *reward* for taking risk:

$$S_{r_f}(w,\mu,\Sigma) = \frac{w^{\mathrm{T}}\mu - r_f}{\sqrt{w^{\mathrm{T}}\Sigma w}}$$
(4.7)

The problem of finding an admissible portfolio that maximizes the reward of taking risk can be posed as the optimization problem:

$$\max_{w} S_{r_f}(w, \mu, \Sigma)$$

$$s.t. \quad 1^{\mathrm{T}}w = 1$$
(4.8)

The optimal value is known as the market price of risk. When the risk free asset is available, the efficient frontier is a line of the form:

$$r = r_f + \left(\max_{\mathbf{1}^T \mathbf{w} = \mathbf{1}} S_{r_f}(w, \mu, \Sigma)\right)\sigma$$
(4.9)



Figure 4.1: Mean Variance Efficient Frontier with and without a risky asset.

When a risk free asset is introduced and combined with any other portfolio of assets, the change in return is linearly related to the change in risk as the weights in the combination vary. The efficient frontier is obtained by maximizing the Sharpe ratio and is shown as a blue line in Figure 4.1 that passes through the risk free asset and is tangent to the efficient frontier of risky assets. In that case, the efficient frontier is known as the Capital Market Line.

The above formulation assumes that the model parameters, (i.e. the assets mean returns and the variability of returns) are known and correct. However, this is hardly the case as in practice the parameters are difficult to estimate, or, are estimated with error. Kim and Boyd (2007) propose the following application of robust control to mean variance analysis and optimization as a mean of finding portfolio weights that perform reasonably well despite estimation error or model uncertainty. Consider the case where the uncertainties in the mean return vector and the covariance are independent of each other. Let The $\mathcal{M} \subseteq$ \mathbb{R}^n denote the set of all possible expected return vectors and $\mathcal{S} \subseteq \mathbb{S}^n_{++}$ the set of all possible covariances where \mathbb{S}^n_{++} is the set of all $n \times n$ symmetric positive definite matrices. Theseparableuncertaintysetcanbewrittenas:

$$\mathcal{U} = \mathcal{M} \times \mathcal{S} \subseteq \mathbb{R}^n \times \mathbb{S}^n_{++} \tag{4.10}$$

In the above formulation the uncertainties in the mean return and covariance are independent of each other. Furthermore, it is assumed that \mathcal{M} and \mathcal{S} are compact (bounded and closed). For a given portfolio w, the expected mean return under parameter uncertainty would lie within bounds:

$$\min_{\mu \in \mathcal{M}} \mathbf{w}^{\mathrm{T}} \mu \le r \le \max_{\mu \in \mathcal{M}} \mathbf{w}^{\mathrm{T}} \mu$$
(4.11)

The worst case (smallest possible) return is then $r_{wc}(w) = \min_{\mu \in \mathcal{M}} w^{T} \mu$. The variance lies within:

$$\min_{\Sigma \in \mathcal{S}} \sqrt{\mathbf{w}^{\mathrm{T}} \Sigma \mathbf{w}} \le \sigma \le \max_{\Sigma \in \mathcal{S}} \sqrt{\mathbf{w}^{\mathrm{T}} \Sigma \mathbf{w}}$$
(4.12)

Similarly the worst case (maximum possible) risk is $\sigma_{wc}(w) = \max_{\Sigma \in S} \sqrt{w^T \Sigma w}$. Assuming that the model uncertainty set \mathcal{U} is connected, the performance of the portfolio lies within a box in the mean variance space, denoting the set of all possible combinations of risk and return the investor might end up with when investing on the portfolio w. Figure 4.2 shows an example of this. The worst-case scenario is the lower right corner (the blue dot) as it has the lowest return with the highest risk.

In the traditional Markowitz formulation, an investor with mean variance preferences who knows the true parameters of his model with certainty and trusts them, would prefer a portfolio w_1 to a portfolio w_2 if w_1 has a higher (or equal) return and a lower (or equal) risk.



Figure 4.2: Return-risk space for three portfolios under separable model uncertainty.

Inspired by the max-min expected utility theory of Gilboa and Schmeidler (1989), Kim and Boyd extent the notion of portfolio preference to *robust mean-variance analysis* with separable model uncertainty; when an individual is considering a specific portfolio (weights) he is considering the set of all possible outcomes. Being uncertainty averse, the decision is made with respect to the least favorable (worst case) outcome. An investor would prefer portfolio w_1 to portfoliow₂ if w_1 has a higher (or equal) worst-case return and a lower (or equal) worst case risk. In Figure 4.2, portfolio w_2 is preferred to w_3 in the worst-case sense, however no clear preference relation can be made between any other two pairs. Under this formulation, the investor would seek portfolio weights that maximize the worst-case mean return as long as the worst-case risk remains within a level of tolerance. The robust counterpart of the portfolio optimization problem can be posed as:

$$\underset{w}{\text{maximize } r_{wc}(w) }$$
(4.13)
s.t. $\sigma_{wc}(w) \le \sigma^{\text{tol}}$
 $1^{\text{T}}w = 1$



Figure 4.3: Robust mean variance frontier. Portfolios w_1 and w_3 are robust mean variance efficient while w_3 is not.

where $r_{wc}(w)$ and $\sigma_{wc}(w)$ denote the worst case mean return and volatility respectively and σ^{tol} is the maximum level of risk that the investor is willing to undertake. For every possible level of risk tolerance $\sigma^{\text{tol}} \in (0, \infty)$ the optimal solution of the problem defines the curve:

$$f_{rob}(\sigma) = \max_{\substack{1^{\mathrm{T}} \mathrm{w}=1, \sigma_{wc}(\mathrm{w}) \le \sigma^{\mathrm{tol}}}} r_{wc}(\mathrm{w})$$
(4.14)

The above curve is defined as the *robust mean variance efficient frontier* and denotes the set of Pareto optimal worst-case performance that can be achieved. A portfolio is called*robust mean variance efficient* if its worst-case risk and return lie on the above curve. TheworstcaseSharperatioisalsodefinedas:

$$S_{wc}(w, r_f) = \min_{\mu \in \mathcal{M}, \Sigma \in \mathcal{S}} S_{r_f}(w, \mu, \Sigma)$$
(4.15)

In the traditional Markowitz formulation, the investor has mean-variance preferences, i.e. his portfolio choice is based only on the mean return and risk of the resulting portfolio.

Furthermore, it is assumed that he is risk averse, i.e. between two portfolios with equal expected mean return, he would choose the one with the less risk. Such a behavior can be modeled with the help of a utility function:

$$u\left(\mathbf{w}^{\mathrm{T}}\mathbf{\mu},\sqrt{\mathbf{w}^{\mathrm{T}}\Sigma\mathbf{w}}\right)$$
(4.16)

Since the investor is risk averse, $u(w^T\mu, \sqrt{w^T\Sigma w})$ is strictly increasing in $w^T\mu$ (for fixed $\sqrt{w^T\Sigma w}$) and strictly decreasing in $\sqrt{w^T\Sigma w}$ (for fixed $w^T\mu$). A commonly used function is the expected quadratic utility function:

$$u = w^{\mathrm{T}} \mu - \frac{\gamma}{2} w^{\mathrm{T}} \Sigma w \tag{4.17}$$

where $\gamma > 0$ is the risk aversion coefficient, denoting the strength of preference towards risk. When the investor trusts the parameters of his model, the portfolio selection problem reduces to maximizing his utility:

$$\max_{w} \left(w^{T} \mu - \frac{\gamma}{2} w^{T} \Sigma w \right)$$
s.t. 1^T w = 1
$$(4.18)$$

In the robust portfolio formulation of Kim and Boyd (2007), investor's preferences towards parameter uncertainty are modeled as a text-book robust control max - min optimization problem. The investor is trying to maximize his utility while a malevolent agent is trying to minimize it through his control on the uncertain parameters. The control of the malevolent agent is constrained by the uncertainty set, denoting the investor's strength towards uncertainty aversion. Theproblemisposedas:

$$\max_{w} \min_{\mu \in \mathcal{M}, \Sigma \in \mathcal{S}} \left(w^{\mathrm{T}} \mu - \frac{\gamma}{2} w^{\mathrm{T}} \Sigma w \right) (4.19)$$

s. t. $1^{T}w = 1$

orequivalentlyas:

$$\max_{\mathbf{w}} \left(r_{wc}(\mathbf{w}) - \frac{\gamma}{2} \sigma_{wc}(\mathbf{w}) \right) (4.20)$$

s.t.
$$1^{T}w = 1$$

Denote by w^{*} the solution to the above problem and U^* the optimal utility under the worstcase scenario. In the risk-return space, the quadratic curve $r = U^* + (\gamma/2)\sigma^2$ is tangential to the robust efficient frontier at portfolio-point w^{*}. This is illustrated at Figure 4.4.



Figure 4.4: Robust efficient frontier and quadratic utility maximization under model uncertainty (without the risk free asset).

Kim and Boyd (2008) extend the two-fund separation theorem demonstrating that an investor with a quadratic utility function as above and uncertainty aversion can separate the investment problem into two steps; first, find the portfolio of risky assets that maximizes the

worst case Sharpe ratio over all possible asset return statistics. Then decide on the mix of the risky portfolio and the risk free asset, depending on the investor's attitude towards risk. Thiscanbewrittenas:

$$\max_{w} \min_{\mu \in \mathcal{M}, \Sigma \in S} \frac{w^{\mathrm{T}} \mu - r_{f}}{\sqrt{w^{\mathrm{T}} \Sigma w}}$$
s.t. 1^Tw = 1
$$(4.21)$$

or, equivalently:

$$\max_{\mathbf{w}} S_{wc}(\mathbf{w}, r_f)$$
(4.22)
s.t. 1^Tw = 1

Once the portfolio w^{*} that maximizes the worst case is obtained, the problem then reduces on deciding the mix between the risk-free asset and w^{*}, based on the risk preferences of the investor. As the risk-free asset has zero variance, the change in return is linearly related to the change in risk as the mix changes. In the mean-risk space, the investing portfolio w^T would lie in the line that connects the risk-free asset and the portfolio w^{*} and is tangent to $r = U^* + (\gamma/2)\sigma^2$ where U^{*} is the optimal utility under the worst case scenario. This can be seen in Figure 4.5.

Finally, Kim and Boyd (2007) present a numerical example with 8 risky assets when short selling is prohibited. They consider the case where the possible variation in the expected return of each asset is at most 20% and the possible variation in the expected return of a uniformly weighted portfolio is at most 10%. Figures 4.6 and 4.7 are obtained from their work and demonstrate the performance of the nominal and robust efficient frontier under the baseline and worst case scenario.



Figure 4.5: Robust efficient frontier and quadratic utility maximization under model uncertainty when the risk free asset is available.



Figure 4.6: Nominal Efficient Frontier and Robust Efficient Frontier under the baseline model.

Source: KimandBoyd (2007)



Figure 4.7: Nominal Efficient Frontier and Robust Efficient Frontier under the worst case scenario.

Source: Kim and Boyd (2007)

Figure 4.6 demonstrates that robust efficient frontier portfolios do not perform as well as the nominal ones under the baseline scenario, i.e. if the point estimates of mean and variance hold exactly. However, as can be seen in Figure 4.7, robust efficient frontier portfolios achieve better performance than the nominal ones under the worst case scenario as they are less sensitive to variations in the parameters. By choosing a portfolio of the robust efficient frontier, the investor is willing to trade a little part of baseline optimality to make sure that the performance of his portfolio is less sensitive to parameter variations and performs relatively better under the worst-case scenario. Finally, Figure 4.8 shows the optimal weight allocations for possible values of volatility. In this example, the most noticeable difference is that robust efficient portfolios are more diversified than nominal ones with the same risk levels, and therefore less likely to produce extreme results.



Figure 4.8: Optimal portfolio weights for possible values of risk. Top: nominal efficient portfolios.Bottom: robust efficient portfolios. Robust efficient portfolios are much more diversified that the nominal ones for the same amount of risk.

Source: Kim and Boyd (2007)

Chapter 5: Conclusions

Uncertainty aversion is a distinct behavior that can have very different implications than risk aversion. Ellsberg (1961) demonstrated through his famous paradox that expected utility theory cannot predict uncertainty related preferences and behavior. Despite this, uncertainty related preferences were not modeled in economics until recently. The most common approaches are owed to Gilboa and Schmeidler (1989) who axiomatized preferences under uncertainty by proposing the max-min expected utility and Hansen, Sargent and other co-authors (1999,2003, 2007) who were inspired by the robust control theory. This dissertation mainly focused on applications of robust control in modeling uncertainty in financial economics and specifically in asset pricing. A common observation is that the market price of risk compensates for both bearing risk as well as bearing uncertainty. Hansen et al. (1999) extend the consumption-based asset-pricing model as to include a parameter that indicates the strength of the decision maker's uncertainty aversion and generates an "uncertainty premium". Maenhout (2004) moves along the same lines clearly distinguishing between risk aversion, uncertainty aversion and intertemporal substitution, and further suggests that uncertainty aversion is environment specific and cannot be measured by stylized experiments with known probabilities. Finally, other applications of robust control can be found in the formulation of robust efficient frontiers. Portfolios that are generated under this approach are less sensitive to estimation errors in the mean and variance of the assets' returns. Although robust efficient frontier portfolios do not perform as well as the nominal ones under the baseline scenario, i.e. if the point estimates of mean and variance hold exactly, their performance is superior under the worstcase scenario.

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