ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

MASTER THESIS

Portfolio Construction

Author :
Maria Amorgianou

Supervisor :
Ioannis Vrontos

DEPARTMENT OF INFORMATION SCIENCES AND TECHNOLOGY DEPARTMENT OF STATISTICS

July 2020

Abstract

The selection of the optimal portfolio is a problem that concerns the financial science. The term "portfolio" refers to the collection of assets owned by an investor. An "optimal portfolio" is a specific mix of assets that optimizes a utility function. Nowadays, due to the development of information technology, we are able to control complex mathematical models and select the optimal portfolio that provides less risk and the highest out-of-sample risk-adjusted realized return. Based on this, many models such as the Single Index model, Static Latent factor model, Latent factor GARCH model, full-factor multivariate GARCH model, regime switching-dynamic correlations model have been developed. This thesis presents a multidimensional methodology based on the financial analysis of hedge fund returns.

Keywords : Portfolio Construction, Dynamic covariances / correlations, Hedge fund portfolios

ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

Περίληψη

Η επιλογή του βέλτιστου χαρτοφυλακίου είναι ένα πρόβλημα που αφορά τις χρηματοοικονομικές επιστήμες. Ο όρος "χαρτοφυλάκιο" αναφέρεται στην συλλογή περιουσιακών στοιχείων που ανήκουν σε έναν επενδυτή. Ένα "βέλτιστο χαρτοφυλάκιο" είναι ένα συγκεκριμένο μείγμα περιουσιακών στοιχείων που βελτιστοποιεί μια συνάρτηση χρησιμότητας. Σήμερα, λόγω της ανάπτυξης της τεχνολογίας των πληροφοριών, είμαστε σε θέση να χειριζόμαστε πολύπλοκα μαθηματικά μοντέλα και να επιλέγουμε το βέλτιστο χαρτοφυλάκιο που παρέχει λιγότερο κίνδυνο και το υψηλότερο έξω από το δείγμα προσαρμοσμένο κινδύνο πραγματοποιόντας απόδοση. Με βάση αυτό, πολλά μοντέλα, όπως το μοντέλο ενός κοινού παράγοντα (SIM), το μοντέλο λανθάνουντα παράγοντα (Static Latent factor model), το λανθάνον μοντέλο (Latent factor) GARCH, το μοντέλο GARCH πολυπαραγοντικού πολλαπλού παράγοντα, το μοντέλο δυναμικής συσχετισμούς που μεταλλάσσουν το καθεστώς έχουν αναπτυχθεί. Αυτή η διπλωματική εργασία παρουσιάζει μια πολυδιάστατη μεθοδολογία που βασίζεται στην χρηματοοικονομική ανάλυση των αποδόσεων των αντισταθμιστικών κεφαλαίων.

Acknowledgments

I would like to seize this opportunity to thank my supervisor Prof. Ioannis Vrontos for his support, guidance, and help throughout the duration of the thesis. I am indebted to him for all the knowledge and insights he shared with me.

I also wish to thank all my friends and family who have provided encouragement and endearing support.

Contents

6 Multivariate Heteroscedasticity models

7 Application to real data

Chapter 1

Introduction

Capital markets and stock exchanges are an important measure of the economic development of an economy which has shown growth in recent years to a large extent. The investors, who are trying to calibrate the cost and risk of investments every time, are constantly seeking ways and products to increase their profit. They want to develop effective portfolios, which are a better combination of risk and returns, so as to minimize the risk, that is the diversification. The diversification of the portfolio is essentially when the investor tries to get rid of any non-systemic risk, i.e. any risk eliminated by the appropriate combination of funds, portfolio and assume only the market risk. Harry Markowitz is the founder of the theory, who developed the corresponding theory in 1952 and won the Nobel Prize for his work in 1990. Markowitz developed the theory of the optimal portfolio based on analysis of variance and covariance between securities in order to maximize expected return and minimize risk. Moreover, Sharpe (1966) calculated the profit that an investor would have through the Sharpe index from a specific distribution of securities. He combined the expected return and standard deviation of portfolios to combination with the risk-free security. Because of the globalization of markets, risk management has been an imperative need as we understand. In fact, the impact of one market on the other is immediate. Thus, the fall in the prices of securities traded in one country could be followed by the fall in the prices of securities in another country. For this reason, the construction of the Mean-variance and Minimum variance portfolio were developed. According to this theory, the investors make the effort to maximize their wealth and to achieve diversification of their portfolio minimizing the potential loss from their investment. In this thesis, we will present and analyze econometric models applying them to financial problems. We will see the construction of optimal portfolios, the assessment of the performance, and the risk of financial capital. We will present financial models that are appropriate for financial data with their specific characteristics. The characteristics of financial data present the phenomenon of fat tails of returns, deviation from normality and variances, and covariance in returns.

In the second chapter, we analyze the portfolio management process, the risk and return of the assets, and portfolio diversification. These concepts are very important for understanding the whole process before making optimal portfolios. The third chapter presents the problem of portfolio construction. From solving a problem of minimizing the variance of the portfolio under certain constraints, we find the optimal investment rate i.e. the weights of the assets. Also important for finding the optimal portfolio is the forecast of expected return and variance of the portfolio. In the fourth chapter, we refer to the equation of the Capital Market Line, the Capital Asset Pricing Model, the Arbitrage Pricing Theory, and performance measure. The use of the models is a useful tool in portfolio management as we receive the returns of the securities in these processes. But also the performance measures are very useful for making a reliable estimate of the returns of investment funds. In the fifth and sixth chapter describe and we develop econometric models that study the relationship between returns of the assets with some multivariate factor models. Furthermore, we apply some mentioned econometric models and techniques to an empirical financial problem.

Chapter 2

Portfolio Theory

2.1 The portfolio management process

The portfolio management process involves a comprehensive dynamic sequence of steps that never stops to create and maintain an appropriate portfolio. The whole process starts with the initial investment of funds based on some plans that the investor has. The real work starts from the evaluation of the performance of the portfolio and the modification of the portfolio based on the needs of the investor and the changes in the financial environment (Reilly F. And Brown K. 2012).

According to Reilly and Brown, this process is divided into a series of four consecutive steps. Initially, the first phase has to do with the policy statement which focuses on the needs of the investor, which may be short-term or long-term and these will be determined by the expectations of capital markets. This phase the investor's target and investment constraints are determined, thus ensuring that the investment decisions are appropriate for the investor. As a result of this phase, the risks become apparent which the investor is willing to take over. The period time plays a very decisive role in shaping the investment objectives, needs, and investment constraints of an investor and should be taken into account. Also, the conditions are distinguished which prevails in the economic climate during this period analysis, as the characteristics and prospects of the various investment trades are configured. At this phase, estimates and forecasts of the capital market expectations are made.

Following the first phase is the construction phase in which the asset allocation of the investor takes place within the diversity of asset classes e.g. investment bond. The asset allocation may have either regular or strategic character into different types of investment products, depending on the short-term or long-term investment policy to be followed by the investor. During the planning phase, it was observed that there are some securities with more attractive features. So, these options are selected for each type of capital investment in the construction phase. The portfolio optimization process is the final step in this phase which is based on the determination of the percentages of capital that will be invested in any bond. In essence, the exact determination of the percentages of capital takes place that will be invested in each bond. This is achieved through the use of mathematical programming. Jacobs and Levy (1995) introduce the terminology of portfolio engineering for this phase.

Then, there is the evaluation phase in which is done the calculation of portfolio performance using risk-adjusted performance measures. This phase is a comparison with the performance of the other portfolios that are considered standard (benchmark portfolios) and with different types of market portfolios. In addition, an audit of discrepancies is carried out concerning the investment objectives and investment constraints which had set by the investor. Namely, it examines how well the planning phase was followed.

Finally, the revision phase follows which may be a change in the choice of securities and the percentages of capital invested in these securities. It is also possible to differentiate the initial decisions concerning asset allocation in various types of investment products. These diversifications can be made because of the change in the prevailing conditionals at the market and changing investor preferences in particular within the investment objectives and investment constraints. There is constant monitoring of the changes either the investor's policy or the market parameters and these are taken into account during the revision phase (Elton, E.J., Gruber, M.J., Brown, S.J. and Goetzmann W.N. 2014).

Portfolios management is considered a recurring process, as reported by Maginn et al. (2007) in which is realized determining investment objectives and constraints, launching investment strategies, designing the composition of the portfolio, evaluating the portfolio performance, continual monitoring the change of its conditions market and the preferences of the investor and redesigning the composition of the portfolio.

2.2 Return and Risk

Several factors and investment characteristics play a decisive role during the process of constructing the optimal portfolio. The most important of these factors are the risk and return to individual assets. Correlations among individual assets along with risk and return are important determinants of portfolio risk. An investor needs to understand the risk and the return of investing in creating a portfolio. The basic idea is that an investor desires to maximize the expected return of this portfolio for a given risk level or to minimize the risk of the portfolio for a particular expected return provided by Elton, Guber, Brown, and Goetzmann (2006). An investor will move from one portfolio to another one which has the same expected return but less risk, or to a portfolio which has the same risk but greater expected return (Fama, E. and French, 1993).

The return of a portfolio is commensurate with the returns of its individual assets, so the return of a portfolio is the weighted average of the returns of its component assets. Suppose that a portfolio has n assets and $R_{i,t}$ is the return of a asset i per year t. In addition, the expected return of a asset i per year t corresponds $\mu_{i,t} = E(R_{i,t})$ and the variance corresponds $\sigma^2_{i,t} = V(R_{i,t})$. The percentage of capital invested in asset i is w_i, $i = 1,....,n$ with $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$, so the return of a portfolio at the t is given by the relationship :

$$R_{p,t} = w_1 R_{1,t} + w_2 R_{2,t} + \ldots\ldots + w_n R_{n,t} = \sum_{i=1}^{n} w_i R_{i,t} = w' \cdot R_t$$

where $w = (w_1, w_2, \ldots, w_n)'$ it is the $n \times 1$ vector of investment rate or weights and $R_t = (R_{1,t}, R_{2,t}, \ldots\ldots R_{n,t})'$ is the $n \times 1$ vector returns at time t. So, the expected return of a portfolio at time t is given by the relationship :

$$E(R_{p,t}) = E(w_1 R_{1,t} + w_2 R_{2,t} + \ldots\ldots\ldots + w_n R_{n,t}) =$$
$$w_1 E(R_{1,t}) + w_2 E(R_{2,t}) + \ldots\ldots + w_n E(R_{n,t}) =$$

$$\sum_{i=1}^{n} w_i E(R_{i,t}) \;=\; \sum_{i=1}^{n} w_i \mu_{i,t} \;=\; w' \cdot \mu_t$$

where $\mu_t = (\mu_{1,t}, \mu_{2,t}, \ldots\ldots, \mu_{n,t})'$ is a $n \times 1$ vector of mean returns of the asset at time t. The variance return of a portfolio at time t is given by the relationship :

$$V(R_{p,t}) = V(w_1 R_{1,t} + w_2 R_{2,t} + \ldots . + w_n R_{n,t}) =$$

$$w_1^2 \cdot V(R_{1,t}) + \ldots . . + w_n^2 \cdot V(R_{n,t}) + 2 \cdot w_1 \cdot w_2 \cdot Cov(R_{1,t}, R_{2,t}) + \ldots . . + 2 \cdot w_{n-1} \cdot w_n \cdot Cov(R_{n-1}, R_{n,t}) =$$

$$\sum_{i=1}^{n} w_i^2 \cdot \sigma_{i,t}^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} w_i \cdot w_j \cdot \sigma_{ij,t} = \sum_{i=1}^{n} w_i^2 \cdot \sigma_{i,t}^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} w_i \cdot w_j \cdot \rho_{ij,t} \cdot \sigma_{i,t} \cdot \sigma_{j,t} =$$

$$w' \cdot \Sigma_t \cdot w = w' \cdot D_t \cdot R_t \cdot D_t \cdot w$$

where :

- $\sigma_{ij,t} = Cov(R_{i,t}, R_{j,t})$ is the covariance between the return $R_{i,t}$ of asset i and the return $R_{j,t}$ of asset j at time t
- $\rho_{ij,t} = Corr(R_{i,t}, R_{j,t})$ is the correlation between the return $R_{i,t}$ of asset i and the return $R_{j,t}$ of asset j at time t
- Σ_t is a $n \times n$ covariance matrix of the returns of assets at time t
- D_t is a $n \times n$ diagonal matrix with element of standard deviation $\sigma_{i,t}$
- R_t is a $n \times n$ correlation matrix of the returns of assets

From the above, it seems that the risk returns of the portfolio are determined by the risk returns of the individual assets and the covariance of the assets but also from the correlation returns of the assets. Covariance is a statistical measure and it declares how one investment moves with another. Observing the factors, which determine the variability of one's return portfolio, we conclude that the higher is the variation of the return of the individual assets, the riskier will become the portfolio. The correlation coefficient measures the degree of correlation ranging from -1 for a perfectly negative correlation to +1 for a perfectly positive correlation. If the correlation coefficient is large, the covariance is also large and therefore the risk of the portfolio. On the contrary, if the correlation coefficient is small, the covariance is small and therefore the risk of the portfolio. If a portfolio has a correlation coefficient close to zero, it will be an uncorrelated investment pair. Note that the correlation coefficient of investments won't have an exact correlation coefficient of zero. Also, the higher is the number of bonds in a portfolio, the lower is the risk while the difference portfolios compositions cause different results that determine the expected return of the portfolio. If there are x securities, infinite combinations can be made to each other and to form infinite portfolios as Wallengren and Sigurdson 2017 mentioned.

The risk can be defined as the deviation from the expected historical returns during a given period time as (Bofah and McClure, 2010) mentioned. However, Markowitz's portfolio selection argues that " The risk of an asset isn't the risk each asset individually, but the contribution each asset at the risk of the general portfolio " as Royal Swedish Academy of Sciences, 1990 mentioned. In a portfolio, the total risk of a title can be divided into two categories which are systemic risk (market risk) and non-systemic risk (specific risk).

The systemic risk or market risk cannot be limited by them, the investors. The economic, social and political conditions that prevail in the local and international environment, such as unemployment. Some risk premium is sought by investors to be protected from systemic risk as Mangram, 2013 and Ross, Westerfield, and Jaffe, 2002 mentioned.

The non-systemic risk or specific risk is related to the risks contained in each stock separately such as for example the fall in the stock price of a bank and this will not affect the prices of other stock owned by other banks. This specialist risk can be addressed through the process of diversification portfolio. If the investors increase the number of stocks consisting of the portfolio, the dispersion of stock is achieved and therefore its diversification portfolio.

2.3 Portfolio Diversification

Portfolio diversification is a risk management technique where investment products are bought with different risk and return, in order to reduce as much as possible the risk taken and to normalize a significant percentage of the variability of the overall return of a portfolio. The effects of both systemic and non-systemic risks can be avoided by diversifying the portfolio and absorbing the losses or increasing the probability of profit that may result from poor returns on the individual investment. A rational investor will make such a distribution of assets that for a given level of risk, the return of the portfolio will be maximized. In other words, there will be no portfolio or individual investment that offers higher returns with the same level risk. The correlation plays an important role. If the investments are uncorrelated with each other, the total investment risk is reduced by investing in different categories of assets and securities of different publishers or even countries. The lower the degree of correlation between investments in a portfolio, the higher the degree of diversification achieved, and therefore the smaller the number of assets needed. In a diversified portfolio, the overall return may be lower but more stable than that of a non-diversified portfolio. There is an inverse relationship between risk and return on investment in an effective market. One of the easiest and most straightforward ways to diversify your portfolio is the purchase stocks together in hedge funds (Woohwan Kim, Young Min Kim, Tae-Hwan Kim and Seungbeom Bang).

.

Types of diversification

Many options for diversifying the portfolio can be used by the investor, in a wide range of cost, return, and risk options. In general, if the portfolio has a high degree of diversification, this has low systemic and non-systemic risk. This happens because the probability decreases that the value of all assets is reduced at the same time, thus reducing the overall variance in the value of the portfolio.

Horizontal diversification	→	The investment in similar financial products constitutes the horizontal diversification of the portfolio.
Vertical diversification	→	The investment in different financial products and assets is the vertical diversification which can be traded even in different markets. This greatly reduces the risk of zeroing the value of a portfolio, even if there is something that affects the whole economy.
Over-diversification	→	Additional investment in a portfolio does not further improve its risk and return where it is called over-diversification. The benefit resulting from the acquisition of the investment is less than the loss of potential profits (diminishing returns).

Supposing that the assets returns are unrelated to each other, so the correlation coefficient and the covariance are zero, that is $\rho_{ij,t} = \sigma_{ij,t} = 0$, having invested the same percentage of the fund in each asset $w_i = 1/n$ for each i, $i = 1,.....,n$, then :

$$V(R_{p,t}) = \sum_{i=1}^{n} (1/n)^2 \sigma_{i,t}^2 = 1/n \sum_{i=1}^{n} \sigma_{i,t}^2 / n \rightarrow 0 \qquad \text{as} \qquad n \rightarrow \infty$$

The number of assets increases and thus the risk of the portfolio diminishes, that is it goes down to zero.

If we consider a portfolio consisting of two assets and the percentage of the fund investing in the first asset is w_1 and the second asset is $w_2 = 1 - w_1$, $with\ w_i \geq 0\ and\ \sum_{i=1}^{n} w_i = 1$, the expected return of the portfolio at time t, is given by :

$$E(R_{p,t}) = w_1 \cdot \mu_{1,t} + w_2 \cdot \mu_{2,t}$$

$$V(R_{p,t}) = w_1^2 \cdot \sigma_{1,t}^2 + w_2^2 \cdot \sigma_{2,t}^2 + 2 \cdot w_1 \cdot w_2 \cdot \rho_{12,t} \cdot \sigma_{1,t} \cdot \sigma_{2,t}$$

where $V(R_{p,t})$ is the variance of the portfolio at the time t for $R_{1,t}$ and $R_{2,t}$.

Still some cases :

- If the returns of the assets are perfectly positive correlated, that is $\rho_{12,t} = 1$, then the variance and the standard deviation are

$$V(R_{p,t}) = (w_1 \cdot \sigma_{1,t} + w_2 \cdot \sigma_{2,t})^2 \quad \text{and} \quad \sigma(R_{p,t}) = w_1 \cdot \sigma_{1,t} + w_2 \cdot \sigma_{2,t}$$

- If the returns of the assets are perfectly negative correlated, that is $\rho_{12,t} = -1$, then the variance and the standard deviation are

$$V(R_{p,t}) = (w_1 \cdot \sigma_{1,t} - w_2 \cdot \sigma_{2,t})^2 \quad \text{and} \quad \sigma(R_{p,t}) = | w_1 \cdot \sigma_{1,t} - w_2 \cdot \sigma_{2,t} |$$

- If the returns of the assets are uncorrelated, that is $\rho_{12,t} = 0$, then the variance and the standard deviation are

$$V(R_{p,t}) = w_1^2 \cdot \sigma_{1,t}^2 + w_2^2 \cdot \sigma_{2,t}^2 \quad \text{and} \quad \sigma(R_{p,t}) = (w_1^2 \cdot \sigma_{1,t}^2 + w_2^2 \cdot \sigma_{2,t}^2)^{\frac{1}{2}}$$

It appears that the risk of the portfolio diminishes when the returns of the assets are uncorrelated or negative correlated.

Chapter 3

Portfolio Construction

3.1 Minimum Variance Portfolio Construction

A portfolio is called a minimum variance when it is a good diversification portfolio. It will make up of assets that alone have a high risk but when stacked the risk is low for the rate of expected return. The portfolio that has the least variance in a given level of expected return is considered an effective portfolio. Markowitz worked on this to create this portfolio with minimum variance. In order to build an effective portfolio, we need to implement an optimization process to determine the weights of assets that minimize the variance of the portfolio and to find constraints (Taras Bodnar, Nestor Parolya and Wolfgang Schmid 2018).

The optimal weights of the minimum variance portfolios can be found by optimizing the following problem :

$$\min_{w} \frac{1}{2} V(R_{p,t}) = min\{\frac{1}{2} w' \Sigma_t w\}$$

$$s.t. \sum_{i=1}^{n} w_i = 1$$

where $w = (w_1, w_2, \ldots\ldots, w_n)'$ and Σ_t is the nxn covariance matrix of the returns at the time t.

Portfolio weights (w_i) can be either positive (long position) or negative (short position). In this problem we minimize the variance taking into account the constraint.

The Lagrange multipliers :

$$\min_{w,\lambda} L(w, \lambda) = min\{\frac{1}{2} w' \Sigma_t w - \lambda(w' I_1 - 1)\}$$

where $w'I_1 = [w_1 \quad w_2 \quad . \quad . \quad . \quad w_n] \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$,

Partial derivatives of $L(w, \lambda)$ with respect of w and λ :

$$\frac{\partial L(w,\lambda)}{\partial w} = \Sigma_t w - \lambda I_1 = 0 \Rightarrow w = \lambda \Sigma_t^{-1} I_1$$

$$\frac{\partial L(w,\lambda)}{\partial \lambda} = w'I_1 - 1 = 0 \Rightarrow (\lambda \Sigma_t^{-1} I_1)' I_1 - 1 = 0 \Rightarrow \lambda I_1' \Sigma_t^{-1} I_1 = 1 \Rightarrow \lambda = \frac{1}{I_1' \Sigma_t^{-1} I_1}$$

Thus, we have $w = \dfrac{\Sigma_t^{-1} I_1}{I_1' \Sigma_t^{-1} I_1}$, where I_1 is a $n x 1$ vector of ones.

With the above relation, we can calculate the composition of the minimum variance portfolio.

Also, a different portfolio, which is a minimum variance portfolio, exists and produces a portfolio with the smallest variance under the constraint that weights are added to the unit $\sum_{i=1}^{n} w_i = 1$. Here ,we can only use portfolio weights with long positions but the specific optimization problem can be solved with computer skills. This is the following :

$$\min_w \frac{1}{2} V(R_{p,t}) = min\{\frac{1}{2} w' \Sigma_t w\}$$

$$w_i \geq 0, \ \sum_{i=1}^{n} w_i = 1$$

3.2 Mean - Variance Portfolio Construction

Markowitz's famous Mean - Variance model is the basis of every classical approach to portfolio management. This model is based on the assumption that the investor maximizes the return and minimizes the risk of his investment. The objective of portfolio optimization is to find a combination of assets that is the weights of each asset in the portfolio which minimize the standard deviation of the portfolio's return for any given level of expected return. The optimization problem usually consists of certain limitations. The first constraint supposes that the sum weights of the portfolio are 1 and the second constraint demand that the weight of each item in a portfolio is not negative, thus it only admits a long position (Hany Fahmy 2019). Taking into account the objective of minimizing portfolio variance for a given level of expected return summarizing as follows :

$$\min_{w} \frac{1}{2} V(R_{p,t}) = min\{\frac{1}{2} w' \Sigma_t w\}$$

$$w_i \geq 0, \ \sum_{i=1}^{n} w_i = 1 \qquad \text{and} \qquad E(R_{p,t}) \geq r_{Target}$$

where this optimization produces a portfolio with the constraint that the expected return of the portfolio will be greater than or equal to some target return (r_{Target}).

Accordingly, we can optimize based on maximizing return as follows :

$$\max_{w} E(R_{p,t}) = max\{\sum_{i=1}^{n} w_i \mu_{i,t}\}$$

$$w_i \geq 0, \ \sum_{i=1}^{n} w_i = 1 \qquad \text{and} \qquad \sigma_p^2 \leq \sigma_0^2$$

Where spwhere σ_p^2 is the variance of the portfolio and σ_0^2 is the variance which we want.

We can also optimize based on the W.Sharpe index to find the portfolio with maximum value of this index as follows :

$$\max_{w} S_p = \frac{R_{p-}R_f}{\sigma_p}$$

$$w_i \geq 0, \sum_{i=1}^{n} w_i = 1$$

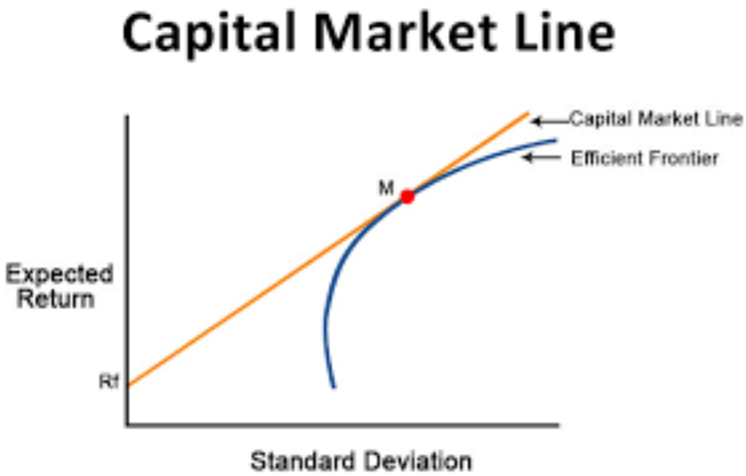
This is a programming problem that can be easily solved using optimization tool. R_p is the expected (mean) return of the portfolio and R_f is the risk free rate.

Chapter 4

Portfolio Evaluation

4.1 The equation of the Capital Market Line (CML)

The Capital Market Line (CML) includes portfolios that combine risk and return. It is the new efficient set replacing Markowitz's efficient set (Salvador Cruz Rambaud, José García Pérez, Miguel Angel Sánchez Granero and Juan Evangelista Trinidad Segovia, 2005).



Supposing that there is a portfolio S, then the blue slope of the line at point S is : $\dfrac{E_{(Rs)} - r_f}{\sigma_s}$

The red slope of the line at point M (the slope of the market line) is : $\dfrac{E_{(RM)} - r_f}{\sigma_M}$

The slopes are equal between them because the two points are in the same line. Therefore :

$$\frac{E_{(Rs)} - r_f}{\sigma_s} = \frac{E_{(RM)} - r_f}{\sigma_M}$$

Thus, the equation of the CML is

$$E(Rs) = r_f + \frac{E(R_M) - r_f}{\sigma_M}\sigma_s$$

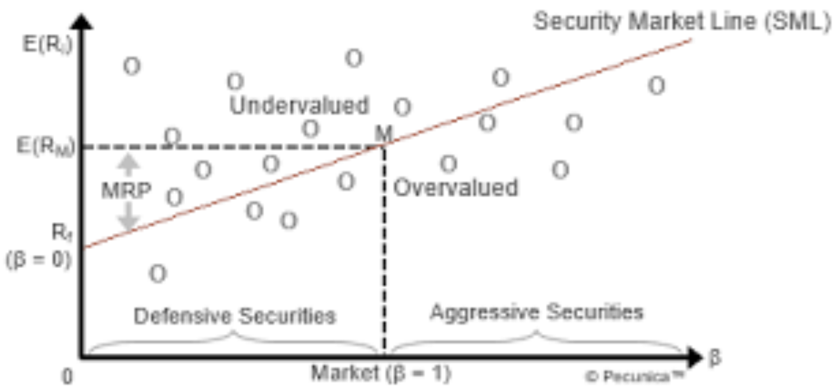
Where

- r_f is the risk free rate
- $\frac{E(R_M) - r_f}{\sigma_M}$ is the slope of the market line (the expected return of the market portfolio).
- σ_s is the standard deviation of portfolio.
- $E(Rs)$ the expected return of the portfolio.

The relationship shows a linear and positive relationship between expected return and risk. The above relationship only applies to efficient portfolios. If we have a stock i, then we use the Capital Asset Pricing Model (CAPM).

4.2 Capital Asset Pricing Model (CAPM)

The first which develop the Capital Asset Pricing Model were William Sharpe (1964), John Lintner (1965), Jan Mossin (1966) and Eugene Fama (1970) in an effort to simplify the Markowitz model and expand it. The basic idea of the model is that the expected return of a security is related to its systemic risk. So based on the CAPM theory, reliable predictions are made with a simple and fast way for its connection expected return of an individual security or one portfolio at market risk (Sharpe, W. F., 1964).



The equation of the model is given by the following relation :

$$(SML) = E(R_s) = E(R_i) = r_f + \beta_i(E(R_M) - r_f)$$

or

$$(SML) = E(R_s) = E(R_i) = r_f + (\frac{E(R_M) - r_f)}{\sigma_M^2})\sigma_{i,M}$$

where

- $E(R_i)$ is the expected return of investment i
- $E(R_M)$ is the expected return of market portfolio
- r_f is the risk free rate
- $\beta_i = \dfrac{\sigma_{i,M}}{\sigma_M^2}$ is security beta, a measure systemic risk
- $\sigma_{i,M}$ is the covariance of the returns of the investment i and the market portfolio
- σ_M^2 is the variance of the market return

In the above equation, the $\beta_i(E(R_M) - r_f)$ is the premium market risk, which depends on the security beta and from the premium market risk. Observing the above relationship we see that the return of a security is positive linear relationship with coefficient β. Namely, it has positive linear relationship between expected return and risk.

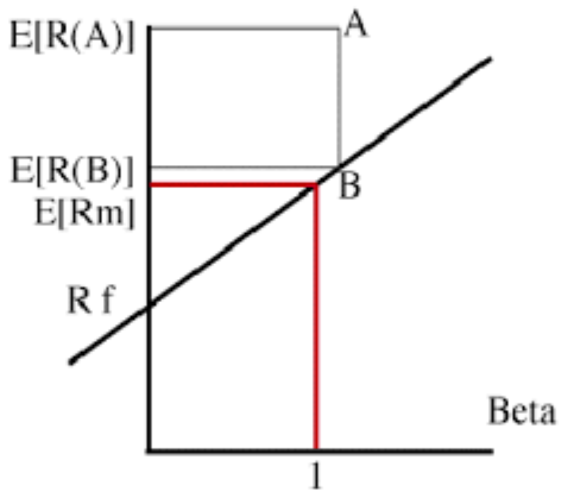
4.3 Arbitrage Pricing Theory (APT)

The model APT (Ross, 1976) is a theory that complements but also contradicts the CAPM. In other words, it is similar to the CAPM, that is, it also refers to the collision of the expected return of a stock with the systemic risk. It includes factors that cannot be eliminated by portfolio diversification and it does not consist of non-system risk. The CAPM deals with the expected return of a stock in relation to the systemic risk of the market while the APT surrounds a large number of factors that affect the expected return. Each factor includes a beta that plays the same role for the market as with the CAPM model. Here the portfolio management policy becomes more completed because we take into account most factors in contrast to the model of the CAPM (Ross, S. A., 1976). According to the APT model, the return of a stock is given by the following relation :

$$E(R_i) = r_f + \beta_{i1} \cdot R_{p_1} + \beta_{i2} \cdot R_{p_2} + \ldots\ldots + \beta_{kn} \cdot R_{p_n}$$

- R_i is the expected return of investment i
- β is the sensitivity of the asset or portfolio in relation to the specified factors
- r_f is the risk-free rate of return
- R_p is the risk premium of the specified factor

That is, the expected return of an asset j is a linear function of the asset's sensitivities to the n factors.



4.4 Performance Measure : Treynor, Sharpe and Janson

If an investment portfolio is considered successful, then the return must be greater than the return on other portfolios in the same category at a given time. In addition to the return, the risk of return must be calculated (Blake, C. R., Elton, E. J., and Gruber, M. J., 1993).

Jack Treynor (1965) was the first who creates a performance evaluation measure. The return generated takes into account the systemic risk. So symbolizing with β_i the systemic risk of the investment i and with r_f the risk free rate, the Treynor performance measure calculated as follows :

$$T_i = \frac{E(R_i)}{\beta_i} = \frac{\overline{R}_i - r_f}{\beta_i}$$

This indicator considers that diversified portfolios only pose a systemic risk β. But, under certain conditions, computational problems can arise which Sharpe tried to solve.

Sharpe (1966) replaced the measure of systemic risk of investment i (β) with the measure of the total risk of investment i (standard deviation of investment returns) as follows :

$$S_i = \frac{E(R_i - r_f)}{\sigma_I} = \frac{\overline{R}_i - r_f}{\sigma_i}$$

The index Sharpe will also include the non-systemic risk if the portfolios are not well diversified. If the differences between the indices Sharpe and Treynor are minimal, then the degree of diversification will have been achieved. The index is suitable for measuring risk at historical values when we have deviations in return that cannot be explained by systemic risk. On the other hand, the index is suitable for predicting future prices.

The indicators Treynor and Sharpe are used for the ranking of portfolios based on returns. If the indicators Treynor and Sharpe are large, the ranking of the return of the portfolio is large. The disadvantages of these indicators are that these use mean values and we cannot make statistical tests to compare these indicators with any other indicator. The index Jenson dealt with these shortcomings.

The Jenson uses historical observations on a macroeconomic level based on the CAPM model. He considered that in order to evaluate the skill of the manager, it is useful to allow the regression equation to have a non-zero constant and we should not limit the regression estimate to be carried out from the beginning of the axes as follows :

$$R_{i,t} - r_f = a_i + \beta_i(R_{M,t} - r_f) + \varepsilon_{i,t}, \varepsilon_{i,t} \sim N(0,\sigma_i^2), i = 1,....,n, t = 1,....,T$$

As we can see from the above equation, part of the return on investment fund is explained by the return of the market index and another by the skill of the manager and the random factor. If the excess return of the index is zero in relation to the financial element without risk, then the constant a_i of the sample shows the expected return on the fund. If it found to be positive and statistically significant, it means that managers achieve higher returns than expected (successful management). If it is low and statistically significant, then we have lower returns than expected (failed management). On the contrary, the portfolio will have achieved exactly the expected return if it is not statistically significant. In the beginning, the investors evaluated the performance of the portfolio based on the degree of return. They knew that there was a risk but they did not know how to measure or quantify it. So, they couldn't take it into account. In the early 1960s, investors learned how to measure risk and factors taking into account separately because until then no measure had been created to combine these ones. There was a grouping of portfolios in which there was similar risk based on some measure and then comparisons were made, based on the return of the grouped portfolios. These performance measures, which we mention, examine risk, and return within an equation.

Chapter 5

Multivariate Multifactor Models

5.1 Introduction

In this section, we will analyze some multi-factor models, thus we will predict the expected return and covariances matrix of financial investments as a function of a limited number of risk attributes. Multivariate multi-factor models are used to construct asset allocation and the risk management of equity portfolios. These can be used to explain individual security or a portfolio of securities. If we use multi-factor models, we will be able to predict the return and to evaluate the variability of the return. One important characteristic of multi-factor models is that these divide the asset returns into common factors and specific factors. However, multi-factor models can be divided into three types : macroeconomic, fundamental and statistical factor models, and these are used to analyze and explain asset prices.

Macroeconomic models :

Macroeconomic models are the simplest models and also, they use observable economic time series. There are macroeconomic variables that are used as factors and these are inflation, the percentage change in industrial production. We call a security's linear sensitivities to the factors as the factor betas of the security. Despite all the previous aspects, there is a minor drawback and it is that they require identification and measurement of all the pervasive sources of risk affecting security returns without knowing exactly what they are.

Fundamental models :

Fundamental models don't use time series regression. These use observed firm or asset-specific attributes as factor betas. Nonetheless, the factor betas are exogenously determined specifically firm-specific attributes where is more the factor returns and are determined empirical random returns associated with these various attributes.

Statistical factor models :

Statistical factor models use unobserved or latent factors. In particular, these models use maximum-likelihood and principal-components-based factor analysis, so produce on the cross-sectional time-series samples of security returns to identify the pervasive factors of risk within returns.

Nevertheless, the portfolios construction has a major problem which in estimating the vector of expected returns and, by extension, in estimating the covariance matrix. Provided that we have a portfolio with n financial assets, then number of parameters is calculated $\frac{n(n+1)}{2}$. Assuming that the estimate of the expected return is n whereas the estimate of the covariance matrix is $\frac{n(n+1)}{2}$. On conditional that we want to confront this dimensionality problem of the factors, so different multivariate multi-factor modes have been created.

5.2 Single Index Model

The single-index model has been developed by William Sharpe (1963) and it is a simple asset pricing model. Moreover, this is based on the hypothesis that the covariance of the financial return is explained through a common single factor, which is the market factor (Lei Huang, Hui Jiang and Huixia Wang, 2019). Also, the single-index model predicts the theoretical returns of the participating stocks. The basic form of the model is given by the relationship :

$$R_{i,t} = a_i + \beta_i \cdot R_{M,t} + \varepsilon_{i,t}, \varepsilon_{i,t} \sim N(0, \sigma_{i,\varepsilon}^2), \quad i = 1,.....,n, \quad t = 1,.....,T$$

From the above equation, the returns of financial assets are unrelated to each other at time t. The expected return and the variance of investment i at time t are written by :

$$E(R_{i,t}) = E(a_i + \beta_i \cdot R_{M,t} + \varepsilon_{i,t}) = a_i + \beta_i \cdot E(R_{M,t})$$
$$V(R_{i,t}) = V(a_i + \beta_i \cdot R_{M,t} + \varepsilon_{i,t}) = \beta_i^2 \cdot V(R_{M,t}) + V(\varepsilon_{i,t}) = \beta_i^2 \cdot \sigma_M^2 + \sigma_{i,\varepsilon}^2$$

where :

- $R_{i,t}$ is the return of investment i at time t
- a_i is the parameter to be estimated and it is independent on the market
- β_i is the parameter to be estimated dependents on the market and determines the return of i stock since there is a change in market return
- $R_{M,t}$ is the market return at time t
- $\varepsilon_{i,t}$ is the innovation term of investment i at time t
- $Cov(\varepsilon_{i,t}, \varepsilon_{j,t}) = 0$ and $E(\varepsilon_{i,t}\varepsilon_{j,t}) = 0$
- $Cov(R_{M,t}, \varepsilon_{j,t}) = 0$

Furthermore, we observe that the expected return and the variance of the returns consist of a systemic component $(E(R_{M,t}), \sigma_M^2)$ and a non-systemic component $(a_i, \sigma_{i,\varepsilon}^2)$. Also, the variance has similarly two parts which are the unique risk (asset specific) $\sigma_{i,\varepsilon}^2$ and systemic risk (index driven) $\beta_i^2\sigma_M^2$.

The covariance of the securities returns $R_{i,t}$ and $R_{j,t}$ is due to only the systemic surge of risk, as the covariance of the securities returns $R_{i,t}$ and $R_{M,t}$ are given by relationships :

$$Cov(R_{i,t}, R_{j,t}) = E[(R_{i,t} - E(R_{i,t}))(R_{j,t} - E(R_{j,t}))]$$

$$= E[(a_i + \beta_i R_{M,t} + \varepsilon_{i,t} - a_i - \beta_i E(R_{M,t}))(a_j + \beta_j R_{M,t} + \varepsilon_{j,t} - a_j - \beta_j E(R_{M,t}))]$$

$$= E[(\beta_i(R_{M,t} - E(R_{M,t})) + \varepsilon_{i,t})(\beta_j(R_{M,t} - E(R_{M,t})) + \varepsilon_{j,t})]$$

$$= E[\beta_i\beta_j(R_{M,t} - E(R_{M,t}))^2 + \beta_i(R_{M,t} - E(R_{M,t}))\varepsilon_{j,t} + \beta_j(R_{M,t} - E(R_{M,t}))\varepsilon_{i,t} + \varepsilon_{i,t}\varepsilon_{j,t}]$$

$$= \beta_i\beta_j\sigma_M^2$$

and

$$Cov(R_{i,t}, R_{M,t}) = \sigma_{iM,t} = E[(R_{i,t} - E(R_{i,t}))(R_{M,t} - E(R_{M,t}))]$$

$$= E[(a_i + \beta_i R_{M,t} + \varepsilon_{i,t} - a_i - \beta_i E(R_{M,t}))(R_{M,t} - E(R_{M,t}))]$$

$$= E[(\beta_i(R_{M,t} - E(R_{M,t})) + \varepsilon_{i,t})(R_{M,t} - E(R_{M,t}))]$$

$$= E[(\beta_i(R_{M,t} - E(R_{M,t}))^2 + \varepsilon_{i,t}(R_{M,t} - E(R_{M,t}))]$$

$$= \beta_i\sigma_M^2$$

From the above relationships, we observe that the covariances depend only on the market factor. Moreover, $\sigma_{i,M} = \beta_i \sigma_M^2 \Rightarrow \beta_i = \dfrac{\sigma_{i,M}}{\sigma_M^2} = \rho_{i,M}\dfrac{\sigma_i}{\sigma_M}$ is called security beta.

- For security beta, $\beta_i = 1$ the returns of the financial element are changed in the same way as the returns of the market factor.
- For security beta, $\beta_i > 1$ the returns of the financial element are changed with greater variation than this returns of the market index. (aggressive financial element)
- For security beta, $\beta_i < 1$ the returns of the financial element are changed with smaller variation than this returns of the market index. (defensive financial element)

If we have a portfolio, then the coefficient α and β are determined as the weighted averages of the individual α and β of the participating stocks. In particular, we have :

Portfolio's a_p : $a_p = \sum_{i=1}^{n} w_i a_i$

Portfolio's β_p : $\beta_p = \sum_{i=1}^{n} w_i \beta_i$

Thus, the expected return of a portfolio and the variance of the returns of a portfolio at time t are given by relationships :

$$E(R_{p,t}) = E(w_1 R_{1,t} + w_2 R_{2,t} + \dots + w_n R_{n,t})$$

$$= w_1 E(R_{1,t}) + w_2 E(R_{2,t}) + \dots + w_n E(R_{n,t}) = \sum_{i=1}^{n} w_i (a_i + \beta_i E(R_{M,t}))$$

$$= \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} w_i \beta_i E(R_{M,t}) = a_p + \beta_p \cdot E(R_{M,t})$$

and

$$V(R_{p,t}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \beta_i \beta_j \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{i,\varepsilon}^2 = (\sum_{i=1}^{n} w_i \beta_i)(\sum_{j=1}^{n} w_j \beta_j)\sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{i,\varepsilon}^2$$

$$= \beta_p^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_{i,\varepsilon}^2$$

In the special case where the weights are equal to each other, that is $\frac{1}{n}$, the equation is :

$$\sigma_p^2 = \beta_p^2 \sigma_M^2 + (\frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} \sigma_{i,\varepsilon}^2)$$

If we have a large number of n within the diversification portfolio, then the second term of the equation tends toward zero and the mean value of the variance of the residuals is equally eliminated. Based on this, the standard deviation of portfolio returns will be given by :

$$\sigma_p = \beta_p \sigma_M = \sigma_M \sum_{i=1}^{n} w_i \beta_i$$

Indeed, we see that the portfolio return variability is determined by the term β_i, as the factors does not affect the volatility of market return σ_M . We define the effect of the non-systemic term as a diversifiable non-systematic risk because this can be eliminated through diversification portfolio or the increase in the average number of participants. On the other hand, the systemic term defines as a no diversifiable systemic risk because it cannot be eliminated through diversification portfolio or the increase in the average number of participants.

Providing that there are n estimates of the a_i , n estimates of the β_i , n estimates of the $\sigma_{i,\varepsilon}$ as well one parameter for the $E(R_{M,t})$ and for the σ_M^2 , then the parameters must be estimated for the portfolio construction is $3n + 2$ when we have n financial elements.

5.3 Multi-Factor Model

The methodologies to estimate multiple factor models are :

1. Time series analysis
2. Cross-section analysis and
3. Statistical factor analysis

The multi-factor model uses the following formula :

$$R_{i,t} = a_i + \beta_{1,i}f_{1,t} + \beta_{2,i}f_{2,t} + \ldots\ldots + \beta_{k,i}f_{k,t} + \varepsilon_{i,t} = a_i + \underset{(1\times k)(k\times 1)}{B_i' f_t} + \varepsilon_{i,t}$$

where :

- $B_i' = (\beta_{1,i}, \ldots\ldots, \beta_{k,i})$ is a $1xk$ vector of the security betas
- $f_t = (f_{1,t}, \ldots\ldots\ldots, f_{k,t})'$ is a $kx1$ vector of the k factors
- ε_i is the specific return to security i

We assume that the f_t is $I(0)$ stationary time series and we also assume that :

- $E(f_t) = \mu_f$ and $\underset{k\times k}{Cov(f_t)} = E[\underset{k\times 1}{(f_t - \mu_f)}\underset{1\times k}{(f_t - \mu_f)'}] = \Omega_f$
- $Cov(f_{k,t}, \varepsilon_{i,t}) = 0$ for all k, i, t, thus the residuals are unrelated to the f_t.
- $Cov(\varepsilon_{i,t}, \varepsilon_{j,s}) = \begin{cases} \sigma_j^2, & \text{for i=j, t=s} \\ 0, & \text{otherwise} \end{cases}$

<u>Cross-section regression model :</u>

This is less intuitive than the time series analysis but this is an equally powerful method. The analyst begins by observing data concerning the sensitivity of stocks to some factors. The prices of the factors, which we are interested in, are calculated based on the returns of the stocks for a certain period of time, which we examine and the sensitivity of the stocks from the specific factors. These estimates are used to calculate the standard deviations and correlations of the factors. The regression is not performed over one stock over all periods but it is performed over a set of stocks in a specific period, then in a subsequent period for the same set of stocks so on. Several periods performed the regression as we want to obtain time series for the factor values. Cross-section data are to a set of observations providing that these taken from different individuals or groups at a single point in time.

So, the cross section regression model is written based on the multi-factor model as following :

$$\underset{N\times 1}{R_t} = \underset{NX1}{a} + \underset{(N\times k)(k\times 1)}{Bf_t} + \underset{N\times 1}{\varepsilon_t}$$

Both the expected return and the variance of the return is computed by :

$$E(R_t) = a + BE(f_t) = a + B\mu_f$$

$$Cov(R_t) = \Omega = BCov(f_t)B' + D = B\Omega_f B' + D$$

where $D = diag(\sigma^2_{i,\varepsilon})$

<u>Time series regression model :</u>

Time series analysis is the most common analysis for the estimated multi-factor models. The analyst assumes that their factors are influenced by stock returns. The recognition of these factors comes from the economic analysis of enterprises. Furthermore, historical data relating to a series of periods are required for the prices of these factors and the returns of the stocks examine after identifying these factors. Thus, the analyst can calculate the sensitivity of stock return, the standard deviation of the factors as well as the correlation among the factors. The accuracy has an important role in the collection of historical data on the values of factors in this method. However, it is not always easy.

The multi-factor model can be written in a time series regression form :

$$\underset{T\times 1}{R_i} = \underset{(T\times 1)(1\times 1)}{1_T a_i} + \underset{(T\times k)(k\times 1)}{FB_i} + \underset{T\times 1}{\varepsilon_i}\ , \quad i + 1,....,N$$

$$\underset{(T\times 1)(1\times T)}{E\ (\varepsilon_i \varepsilon_i')} = \sigma_i^2 I$$

where $I = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{Tx1}$

Factor analysis :

The analyst does not know either the prices of the factor or the sensitivity of the stocks regarding these factors. A set of information is used on the historical returns of some stocks and thus, the method tries to identify one or more statistically significant factors. In fact, only historical returns are used for the structure of the model based on this analysis. Nonetheless, it does not recognize which economic variables the factors represent and this is a great weakness.

5.4 Macroeconomics Model

The fluctuation of the return of the stocks is explained by some variables that affect it. Each financial market has specificities where are observed in the economic structure of each country and in the sectors on which it depends. It is very reasonable for the financial market that the financial market affected by other producers such as factors affecting the world economy and the movement of other financial markets. Macroeconomic factors influence the formation of the stocks and are related to the predictability data in the prices of the stocks (Jaqueson K. Galimberti, 2019).

The factors are directly observable and we use the regression of time series for each stock where we calculate the sensitivities of the stocks in these factors since we have initially selected the risk factors. Prices are the same for all stocks in any given period of time . We use a stratified regression in the returns of the stocks on the sensitivities in these factors many times in order to find the valuation of macroeconomic factors. The factors affect the whole market and thus, these alter the returns of the stocks as well as our estimates for these returns. An additional feature of macroeconomic factors is that the factors are unrelated to the residuals of the model.

The basic macroeconomic models is :

- Sharpe (1970) - Single factor model
- Chen, Roll and Ross (1986) - Multi-factor model

<u>Sharpe (1970) - Single factor model :</u>

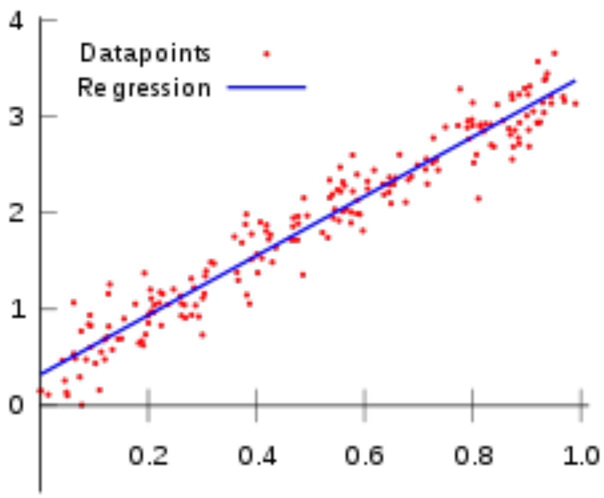
Sharpe assumed that the market index is linearly related to the stock prices. This relationship could be used to estimate return of the stock. We calculate the risk and the return of a stock with this model. Based on the Sharpe model, the sensitivity factors are limited to being positive and concentrating on the unit. The market index is linearly related to the stock prices. This relationship could be used to estimate return of the stock. The risk is calculated by the rate factor beta which calculates the co-variance between the market portfolio and the stock price.

Consider the Sharpe's model (Single Index Model) :

$$R_{i,t} = a_i + \beta_i \cdot R_{M,t} + \varepsilon_{i,t}, \qquad i = 1,....,N, \quad t = 1,.....T$$

where :

$$f_{1,t} = R_{M,t}, \quad \beta_{k,i} = 0, \quad i = 1,.....,N, \quad k = 2,.....K$$



The Sharpe's model has the covariance matrix is given by :

$$\underset{N\times N}{\Omega} = \sigma_M^2 \cdot \underset{(N\times 1)(1\times N)}{B \quad B'} + D$$

where :

$$\sigma_M^2 = Var(R_{M,t}), \quad B = (\beta_1, \ldots, \beta_N)' = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_N \end{bmatrix}, \quad D = diag(\sigma_i^2), \quad \sigma_i^2 = Var(\varepsilon_i),$$

Provided that we will appreciate the β_i and the σ_i^2 from regressing time series

$$\hat{\sigma}_M^2 = \frac{1}{T-1} \sum_{t=1}^{T} (R_{M,t} - \overline{R}_M)^2$$

$$\overline{R}_M = \frac{\sum_{t=1}^{T} R_{M,t}}{T}$$

$$R_{i,t} = \hat{a}_i + \hat{\beta}_i R_{M,t} + \hat{\varepsilon}_{i,t} \Rightarrow \hat{\varepsilon}_{i,t} = R_{i,t} - \hat{a}_i - \hat{\beta}_i R_{M,t}$$

$$\hat{\sigma}_i^2 = \frac{\hat{\varepsilon}_i' \hat{\varepsilon}_i}{T-2}$$

$$\hat{\Omega} = \hat{\sigma}_M^2 \hat{B} \hat{B}' + \hat{D}$$

<u>Multivariate regression :</u>

Multivariate regression analysis is used to predict the value of the stocks and it consist of more responses from a set of predictors. Thus, we see the linear association between more than one predictor and more than one response.

We can now formulate the multivariate multiple regression model as follows :

$$\underset{T\times N}{R_T} = X \cdot \Gamma' + \underset{T\times N}{E_T}$$

where:

$$\underset{T\times 2}{X} = (1 \vdots R_M), \quad \underset{2\times N}{\Gamma'} = (\alpha \vdots B)$$

We estimate the regression coefficients using least squares as follows :

$$\hat{\Gamma}' = (X'X)^{-1}X'R_T$$

The MLE of the covariance matrix Σ is given by :

$$\hat{\Sigma} = \frac{1}{T-2}\hat{E}_T'\hat{E}_T = \frac{1}{T-2}(R_T - X\hat{\Gamma})'(R_T - X\hat{\Gamma})$$

$$\hat{R}_T = X \cdot \hat{\Gamma}' = X(X'X)^{-1}X'R_T$$

Assume that the residuals matrix is given by :

$$\hat{E}_T = R_T - X\hat{\Gamma}' = R_T - X(X'X)^{-1}X'R_T = R_T[I - X(X'X)^{-1}X']$$

<u>Chen, Roll and Ross (1986) - Multi-factor model :</u>

Chen, Roll and Ross (1986) produced multifactorial models which use combinations indicators of macroeconomics factors. Several economic variables are found to be significant which explain expected stock returns. The multi-factor model has k observed macroeconomic variables which are factors f_t. These variables have mean zero and standard deviation one.

The covariance matrix is formulated as :

$$\Omega = B\Omega_f B' + D$$

where :

$$B = (\beta_1, \beta_2, \ldots\ldots, \beta_N)', \quad \Omega_f = E[(f_t - \mu_t)(f_t - \mu_t)']$$

The estimation covariance matrix is given by :

$$\hat{\Omega} = \hat{B}\hat{\Omega}_f\hat{B}' + \hat{D}$$

$$R_i = \hat{a}_i 1 + F\hat{B}_i + \hat{\varepsilon}_i \Rightarrow \hat{\varepsilon}_i = R_i - \hat{a}_i - F\hat{B}_i$$

$$\hat{\sigma}_i^2 = \frac{\hat{\varepsilon}_i'\hat{\varepsilon}_i}{T - k - 1}$$

The estimation of the covariance matrix is formulated as :

$$\hat{\Omega}_f = \frac{1}{T-1}\sum_{t-1}^{T}(f_t - \bar{f})(f_t - \bar{f})'$$

where : $$\bar{f} = \frac{\sum_{t=1}^{T} f_t}{T}$$

5.5 Latent factor model

The factor analysis is the oldest and most well-known statistical method to see the relationship involved in observable and latent variables. This examines the covariance in the observable variables in order to gather information for these factors. A time-series process is followed by the latent factors considered as a vector autoregression (A. J. O'Malley, B. H. Neelon, 2014).

The dynamic factor model is given by :

$$\underset{N\times 1}{X_t} = \underset{N\times q}{\lambda(L)}\ \underset{q\times 1}{f_t} + \underset{N\times 1}{\varepsilon_t}$$

or

$$\underset{q\times 1}{f_t} = \underset{q\times q}{\Psi(L)} f_{t-1} + \underset{q\times 1}{\eta_t}$$

Assuming that the above equations are stationary.

where :

- $\lambda_i(L)$ is the dynamic factor loading for i series
- $\lambda_i(L)f_t$ is the common component of the i series

Static latent factor model :

This model is given by :

$$y_t = \underset{n\times 1}{\mu} + \underset{(n\times K)(K\times 1)}{LF_t} + \varepsilon_t$$

where :

- μ is vector of constants
- L is matrix of factor loading
- $F_t = (f_{1,t}, \ldots\ldots, f_{K,t})'$ is the vector of unobserved common factors

The covariance matrix is formulated as :

$$V^{SLF} = L\Omega^{SLF}L' + D$$

Latent factor GARCH model :

This model reduce the dimension of financial returns which allows for a great flexibility in the econometric specification and in the modeling strategy. The latent factor GARCH model assumes that factors follow a GARCH process.

This model is given by :

$$y_t = \underset{n\times 1}{\mu} + \underset{(n\times K)(K\times 1)}{LF_t} + \varepsilon_t$$

$$\sigma_{k,t}^2 = a_k + b_k f_{k,t-1}^2 + g_k \sigma_{k,t-1}^2$$

The above equation is the variance of the k common factor at time t.

The covariance matrix of asset return at time t is given by :

$$V_t^{LFG} = L\Omega_t^{LFG}L' + D$$

where the diagonal covariance matrix with GARCH variances is Ω_t^{LFG} .

Chapter 6

Multivariate Heteroscedasticity Models

6.1 Introduction

The integration of the financial markets has accelerated due to the economic globalization in recent years. Financial markets are now much more dependent on each other since the prices of one market are spread to another. In that case, this requires a joint study of the markets. As a result, we need to assess the whole economy, when someone has a portfolio the yields of individual assets are considered to be interrelated. Multivariate modeling is similar to single variable modeling. The multivariate models have two problems which are for positive definiteness of the covariance matrix and the number of parameters should be estimated for the assets.

Multivariate heteroscedasticity models have two main problems, one is a large number of parameters to be evaluated and the other is in the difficulty of estimating the covariance matrix which must be positively defined.

Assuming that we have data of the form :

$$y_t \quad , \qquad t = 1,......,T$$

where each $y_t = (y_{1,t}, \ldots . . y_{N,t})$ is a $N \times 1$ vector and providing that the information is until the time t-1, the equation of the mean and the reserved distribution of random errors are given by :

$$y_t = \mu + \varepsilon_t \quad \text{and} \quad \varepsilon_t \,|\, \Phi_{t-1} \sim N_N(0,\Sigma_t)$$

where :

- μ is a $N \times 1$ vector of constants
- ε_t is a $N \times 1$ vector with random errors
- Φ_{t-1} is the set of information at time t-1
- Σ_t is the $N \times N$ covariance matrix with elements $\sigma^2_{i,t}$ and $\sigma_{ij,t}$, $i = 1,.....,N$, $j = i + 1,.....,N$
- $\sigma^2_{i,t}$ is the variance of the i variables at time t
- $\sigma_{ij,t}$ is the covariance between the i and j variable at time t

6.2 Multivariate ARCH model

The financial series were noticed that they show strong asymmetric, curvature but also large and small values of the residuals tend to present within volatility clustering. There were no such models that address these problems. Engle (1982) was studying the inflation in Great Britain observed such characteristics. This demonstrated that variance has a type of heteroscedasticity and this depends on the previous values of the disruptive term (conditional variability). Thus, autoregressive conditional heteroscedasticity models such as ARCH models were created to address these problems (Diaa Noureldin, Neil Shephard and Kevin Sheppard, 2014).

The variance of the disruptive term modifies in the ARCH model over time. So, we observe heteroscedasticity because of the variance depends on the changeability of the previous values. The multivariate ARCH model is given by :

$$y_t = \mu + \varepsilon_t$$
$$\varepsilon_t = \Sigma^{1/2} u_t$$
$$\varepsilon_t \,|\, \Phi_{t-1} \sim N_n(0, \Sigma_t)$$

Where u_t is a k-dimensional white noise and Σ_t is a $N \times N$ conditional covariance matrix of ε_t.

The residuals allocate independently with mean 0 and the constant standard deviation Σ_t. Therefore, we have the conditional covariance matrix calculate as :

$$vech(\Sigma_t) = \gamma_0 + \Gamma_1 vech(\varepsilon_{t-1}\varepsilon'_{t-1}) + \ldots.. + \Gamma_k vech(\varepsilon_{t-1}\varepsilon'_{t-k}) = \gamma_0 + \sum_{i=1}^{p} \Gamma_i vech(\varepsilon_{t-1}\varepsilon'_{t-1})$$

where :

- γ_0 is a $\dfrac{N(N+1)}{2} \times 1$ vector of constants.
- Γ_i is a $\dfrac{N(N+1)}{2} \times \dfrac{N(N+1)}{2}$ coefficient matrices for $i = 1,....,p$.
- vech(.) denotes the operator which inserts the diagonal and lower triangular elements of a symmetrical table in a column table.

Assuming that we consider a bivariate, that is $N = 2$ times series, ARCH(1) process then :

$$vech(\Sigma_t) = \gamma_0 + \Gamma_1 vech(\varepsilon_{t-1}\varepsilon'_{t-1}) =$$

$$vech \begin{bmatrix} \sigma^2_{11,t} & \sigma_{21,t} \\ \sigma_{12,t} & \sigma^2_{22,t} \end{bmatrix} = \begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{21,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \times \begin{bmatrix} \varepsilon^2_{1,t-1} \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon^2_{2,t-1} \end{bmatrix}$$

We have some restrictions on the elements of the γ_0 matrices both the column and the row of the Γ_i because of the conditional covariance matrix must be positively defined at any time and for any ε_{t-1}. Thus, the restrictions are as follows :

$$\gamma_{10} > 0, \gamma_{30} > 0, \gamma_{10}\gamma_{30} - \gamma^2_{20} > 0$$

$$\gamma_{11} \geq 0, \gamma_{13} \geq 0, \gamma_{11}\gamma_{13} - 1/4\gamma^2_{12} \geq 0$$

$$\gamma_{31} \geq 0, \gamma_{33} \geq 0, \gamma_{31}\gamma_{33} - 1/4\gamma^2_{32} \geq 0$$

$$\gamma_{11}\gamma_{33} - \gamma_{22}^2 \geq 0, \gamma_{11}\gamma_{31} - \gamma_{21}^2 \geq 0, \gamma_{13}\gamma_{33} - \gamma_{23}^2 \geq 0$$

We have corresponding restrictions on the rows and columns from the table Γ_1 for multivariate systems larger dimensions than $N = 2$. The $\sigma_{i,t}^2$ and $\sigma_{ij,t}$ depend on the squares of the previous residuals and the cross product all variables of the system.

On the other hand, we can assume that have the diagonal representation. In others words, it provides that each element of the covariance matrix $\sigma_{ij,t}$ is only a function of past values of itself and past values of $\varepsilon_{i,t}$ and $\varepsilon_{k,t}$. So, the bivariate model is defined by :

$$y_t = \beta + \varepsilon_t$$
$$\varepsilon_t | \Phi_{t-1} \sim N_n(0, \Sigma_t)$$

where $y_t = (y_{1,t}, y_{2,t})'$ both $\varepsilon_t = (\varepsilon_{1,t}, \varepsilon_{2,t})'$ with $E(\varepsilon_t) = 0$ and the intercept vector $\beta = (\beta_{10}, \beta_{20})'$ which be known. The variance of the diagonal structure following an $ARCH(1)$ process is defined by :

$$vech \begin{bmatrix} \sigma_{11,t}^2 & \sigma_{12,t} \\ \sigma_{21,t} & \sigma_{22,t}^2 \end{bmatrix} = \begin{bmatrix} \sigma_{1,t}^2 \\ \sigma_{21,t} \\ \sigma_{2,t}^2 \end{bmatrix} = \begin{bmatrix} \gamma_{10} \\ \gamma_{20} \\ \gamma_{30} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t-1}^2 \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{2,t-1}^2 \end{bmatrix}$$

The Σ_t must be positively defined, in that case of the restrictions are as follows :

$$\gamma_{20} > 0, \gamma_{30} > 0, \gamma_{10}\gamma_{30} - \gamma_{20}^2 > 0$$

$$\gamma_{11} \geq 0, \gamma_{22} \geq 0, \gamma_{31}\gamma_{22} - \gamma_{21}^2 \geq 0$$

6.3 Multivariate GARCH

The multivariate GARCH model takes into account the possible correlation that may exist between the economic variables, that is, the influence of one value of the variable on the values of the other variables. The multivariate GARCH model will lead based on recognizing this feature to more empirical models than working with univariate models. Many types of multivariate GARCH models have been studied from time to time (Vrontos, I.D., P. Dellaportas and Politis, D.N., 2003). This section will be developed the Vech, the diagonal Vech, and finally the BEKK models. In particular, the multivariate GARCH model is written :

$$y_t = \mu_t + \varepsilon_t$$

$$\varepsilon_t \,|\, \Phi_{t-1} \sim N_n(0,H_t)$$

6.3.1 VECH - GARCH & Diagonal VECH models

A general form of the VECH model has been proposed by Bollerslev, Engle and Wooldridge (1988) and is given by the following type :

$$vech(H_t) = C + \sum_{i=1}^{p} A_i vech(\varepsilon_{t-1}\varepsilon'_{t-1}) + \sum_{j=1}^{q} B_j vech(H_{t-j})$$

where :

- H_t is the $N \times N$ covariance matrix at time t
- vech(.) denotes the operator which inserts the diagonal and lower triangular elements of a symmetrical table in a column table

- C is a $\frac{N(N+1)}{2} \times 1$ vector
- A_i and B_j are $\frac{N(N+1)}{2} \times \frac{N(N+1)}{2}$ matrices for $i = 1,......,p$ and $j = 1,......,q$

In that case, we use $\frac{N(N+1)}{2} + (p+q)(\frac{N(N+1)}{2})^2$ total number of parameters.

Assuming that we want to evaluate the relationship of the time series between the past value of ones and the current value of the other, we use the bivariate model as follows :

$$vech(H_t) = C + A_1 vech(\varepsilon_{t-1}\varepsilon'_{t-1}) + B_1 vech(H_{t-1}) =$$

$$vech\begin{bmatrix} \sigma^2_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma^2_{22,t} \end{bmatrix} = \begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{21,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \varepsilon^2_{1,t-1} \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon^2_{2,t-1} \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{21,t} \\ \sigma^2_{2,t} \end{bmatrix}$$

This model contains a large number of parameters and is very general. Because of this, we will present the diagonal vector model which consider more successful than the vector GARCH model (Federico Poloni and Giacomo Sbrana, 2014). The diagonal vector model makes the terms, so as it will ensure that the conditional covariance is positively defined. The covariances specifically depend only on the previous cross products and the previous covariances both the variances depend only on its previous squared residuals and the previous variances. Under those circumstances, the number of the parameters appears to be $\frac{N(N+1)}{2}(p+q+1)$. Also, the options for A_i and B_j are limited due to the fact that they can only be diagonal matrices. Therefore, we have the conditional covariance matrix for $N = 2$ and $p = q = 1$ to be given by the following relationship :

$$vech(H_t) = C + A_1 vech(\varepsilon_{t-1}\varepsilon'_{t-1}) + B_1 vech(H_{t-1}) =$$

$$vech\begin{bmatrix} \sigma^2_{11,t} & \sigma_{12,t} \\ \sigma_{21,t} & \sigma^2_{22,t} \end{bmatrix} = \begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{21,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} \varepsilon^2_{1,t-1} \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon^2_{2,t-1} \end{bmatrix} + \begin{bmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{22} & 0 \\ 0 & 0 & \beta_{33} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{21,t} \\ \sigma^2_{2,t} \end{bmatrix}$$

The no conditional covariance matrix is given by :

$$E[vech(\varepsilon_{t-1}\varepsilon'_{t-1})] = [1 - A_1 - B_1]^{-1}C$$

The H_t must be positively defined but the restrictions are difficult to overcome during the assessment process.

6.3.2 BEKK - GARCH & Diagonal BEKK models

The multivariate BEKK model is an advanced form of the multivariate VECH model. This model have a big limitation that it contains a large number of parameters, even in series of small numbers. The classic process of calculating the probability of false-maximum probability is very difficult to assess. Thus, someone can configure or modify the BEKK model using fewer parameters and at the same time someone can turn toothed calculation methods. This model got its name from the initials of the names of the Baba, Engle, Kraft and Kroner whose it was first used (Farid Boussama, Florian Fuchs, and Robert Stelzer, 2011). The main reasoning is that the conditional covariance H_t is positively defined. The diagonal BEKK which we assume the matrices a_{ki} and b_{ki} are diagonal, the scalar BEKK model which is most limited form of the diagonal BEKK, in which have $a_{ki} = aI$ and $b_{ki} = bI$where the a and b are the graded quantities are some examples of reducing the numbers of parameters in the optimization process for the BEKK model.

The Engle and the Kroner (1995) created a BEKK GARCH(p,q) model which the covariance has the following form :

$$H_t = C_0 C_0^T + \sum_{k=1}^{K} \sum_{i=1}^{q} a_{ki}^T \varepsilon_{t-i} \varepsilon_{t-i}^T a_{ki} + \sum_{k=1}^{K} \sum_{j=1}^{p} b_{ki}^T H_{t-j} b_{ki}$$

where :

- C_0 is upper triangular matrix of parameters
- a_{ki} and b_{ki} are $N \times N$ matrices of parameters
- $\varepsilon_{t-i}\varepsilon_{t-i}^T$ is $N \times N$ matrix of the individual errors
- $\varepsilon_t \sim N(0, H_t)$

The positively defined H_t is guaranteed due to the special form taken by the right member of the relationship. The sum of the parameters is $\frac{5N^2+N}{2}$ where N is the number of the assets. The model is possible when the number of the assets is small while on the contrary it becomes disobedient as the number of the assets increases. The greater the number of parameters and securities is more likely that we have negative elements in matrices and therefore, we have negative variances.The elements of the H_t depend from all the resulting diagonals. If we have bivariate BEKK model with $K = 1, p = 1, q = 0$, then the conditional covariance can be written as :

$$\begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{12,t} & h_{22,t} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}' +$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t}^2 & \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} & \varepsilon_{2,t}^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}' +$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} h_{11,t-1} & h_{12,t-1} \\ h_{21,t-1} & h_{22,t-1} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}'$$

Also, the conditional variance of the first asset can be written as :

$$h_{11,t} = c_{11}^2 + a_{11}^2\varepsilon_{1t}^2 + a_{12}^2\varepsilon_{2t}^2 + 2a_{11}a_{12}\varepsilon_{1t}\varepsilon_{2t}$$

And the conditional covariance can be written as :

$$h_{12,t} = c_{11}c_{21} + a_{11}a_{21}\varepsilon_{1t}^2 + a_{12}a_{22}\varepsilon_{2t}^2 + (a_{12}a_{21} + a_{11}a_{22})\varepsilon_{1t}\varepsilon_{2t}$$

These both the rest covariances and variances depend on both the yield squares and the yield diagonals that arise.

The diagonal BEKK model puts the restriction that the a and b are diagonal matrices. In this way, it faces the problem that we have due to the large number of parameters. In fact, the non-negativity of the matrices is more easily ensured. The covariance of the diagonal BEKK GARCH(1,1) model is given by :

$$\begin{bmatrix} h_{11,t} & h_{12,t} \\ h_{21,t} & h_{22,t} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}' + \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t}^2 & \varepsilon_{1,t-1}\varepsilon_{2,t-1} \\ \varepsilon_{1,t-1}\varepsilon_{2,t-1} & \varepsilon_{2,t}^2 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}' +$$

$$\begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} h_{11,t-1} & h_{12,t-1} \\ h_{21,t-1} & h_{22,t-1} \end{bmatrix} \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}'$$

Therefore, the conditional covariance equation are represented as :

$$h_{11,t} = c_{11}^2 + a_{11}^2 \varepsilon_{1,t-1}^2 + b_{11}^2 h_{11,t-1}$$

$$h_{12,t} = c_{12}^2 + a_{11} a_{12} \varepsilon_{1,t-1} \varepsilon_{2,t-1} + b_{11} b_{22} h_{12,t-1}$$

The triangular matrix contain a single element (scalar) so we assume that the coefficient is the same for all our securities during the calculation of the covariance and variance. Therefore, The Ding and Engle (2001) construct the Scalar BEKK model GARCH(1,1), which is given as :

$$H_t = C_0 C_0^T + a\,\varepsilon_{t-1}\varepsilon_{t-1}^T + bH_{t-1}$$

6.4 Constant conditional correlation model

The constant conditional correlation model, as Bollerslev, 1990 mentioned, uses a different approach than the BEKK or the vector GARCH model. This model separates conditional covariance, in k conditional variances but fixed conditional correlation. This model has the advantage that it greatly reduces the parameters and thus calculation process becomes simpler (Begoña Fernández and Nelson Muriel, 2009). Thus, the conditional variance matrix is given by :

$$y_t = \mu_t + \varepsilon_t$$

$$\varepsilon_t | \Phi_{t-1} \sim N_n(0, H_t)$$

$$H_t = D_t R D_t (= \rho_{ij}\sqrt{\sigma_{i,t}^2 \sigma_{j,t}^2})$$

where R is a $N \times N$ correlation matrix, fixed time, with element ρ_{ij} , $i = 1,.....,N,$ $j = 1,.....,N,$ $i \neq j$, and $D_t = diag(\sigma_{1,t} \ldots . \sigma_{N,t})$ is a $N \times N$ diagonal matrix with time varying standard deviations which consist of the conditional standard deviation of the i asset of the i diagonal positioning. Therefore it will be of the form :

$$D_t = \begin{bmatrix} \sigma_{1,t} & 0 & 0 & \dots & 0 \\ 0 & \sigma_{2,t} & 0 & \dots & 0 \\ 0 & 0 & \sigma_{3,t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_{N,t} \end{bmatrix} \quad \text{where} \quad \sigma_{i,t} = \sqrt{\sigma_{ii,t}^2}$$

The constant conditional correlation is given by :

$$R = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \dots & \rho_{1k} \\ \rho_{12} & 1 & \rho_{23} & \dots & \rho_{2k} \\ \rho_{13} & \rho_{23} & 1 & \dots & \rho_{3k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1k} & \rho_{2k} & \rho_{3k} & \dots & 1 \end{bmatrix}$$

The variances $\sigma_{i,t}^2$ follow single variables GARCH(p,q) model :

$$\sigma_{i,t}^2 = a_{0i} + \sum_{j=1}^{p} a_{ij}\varepsilon_{i,t-j}^2 + \sum_{j=1}^{q} \beta_{ij}\sigma_{i,t-j}^2 \quad , \qquad i = 1,.....,N$$

The number of the parameters is equal with $\dfrac{N(N+1)}{2} + N(1+p+q)$ which consist of the ρ_{ij} , a_{0i} , a_{ij} and β_{ij} . The variance matrix H_t is positive definiteness if the constant conditional correlation matrix is positive definiteness and the conditional variances is positive definiteness. So, this is :

$$H_t = \begin{bmatrix} \sigma_{11,t} & \rho_{12}\sigma_{1,t}\sigma_{2,t} & \rho_{13}\sigma_{1,t}\sigma_{3,t} & \cdots & \rho_{1k}\sigma_{1,t}\sigma_{k,t} \\ \rho_{12}\sigma_{1,t}\sigma_{2,t} & \sigma_{22,t} & \rho_{23}\sigma_{2,t}\sigma_{3,t} & \cdots & \rho_{2k}\sigma_{2,t}\sigma_{k,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1k}\sigma_{1,t}\sigma_{k,t} & \rho_{2k}\sigma_{2,t}\sigma_{k,t} & \rho_{3k}\sigma_{2,t}\sigma_{k,t} & \cdots & \sigma_{NN,t} \end{bmatrix}$$

When the CCC model follow GARCH(1,1) and for $N = 2$ then :

$$\sigma_{i,t}^2 = a_{0i} + a_{i1}\varepsilon_{i,t-1}^2 + \beta_{i1}\sigma_{i,t-1}^2 \quad , \quad i = 1{,}2$$

$$\sigma_{12,t} = \rho_{12}(\sigma_{1,t}^2\sigma_{2,t}^2)^{\frac{1}{2}} \quad , \quad -1 \leq \rho_{12} \leq 1$$

In that case, if the H_t is positive definiteness, the ρ_{12} , a_{0i} , a_{i1} and β_{ij} will be $a_{0i} > 0$, $a_{i1} \geq 0$, $\beta_{ij} \geq 0$, $i = 1{,}2$ and $-1 \leq \rho_{12} \leq 1$. Also, the $a_{i1} + \beta_{i1}$ will be $a_{i1} + \beta_{i1} < 1$ for $i = 1{,}2$ so that the variance is finite and there is stationarity.

The assumption of the constant condition correlation is reasonable in some cases. Nevertheless, it remains restrictive because the financial data often see a change of the condition correlation of the financial returns over time. If the period has high variability, the correlation between markets tend to increase.

6.5 Dynamic conditional correlation model

Many scientists consider to be strong the assumption of constant correlations but it does not correspond to reality. So, the DCC model extends the CCC model and the DCC model differs only in allowing R to be time varying (Engle, R.F., 2002).

The estimate of the Engle (2002) consists of two steps :

1. The estimate of the single variable model GARCH and
2. The estimate of the conditional correlation which changes over time The dynamic conditional correlation model (DCC) is defined as :

$$
\begin{aligned}
y_t &= \mu_t + a_t \\
a_t &= H_t^{1/2}\varepsilon_t
\end{aligned}
$$

$$\varepsilon_t \,|\, \Phi_{t-1} \sim N_n(0, H_t)$$

where :

- $H_t = D_t R_t D_t$
- $R_t = (diag(Q_t))^{-1/2} Q_t (diag(Q_t))^{-1/2}$
- $D_t = diag(\sqrt{h_{1,t}}, \sqrt{h_{2,t}}, \ldots, \sqrt{h_{n,T}})$

The y_t is $n \times 1$ vector of log returns of n assets at time t, a_t is $n \times 1$ vector of mean-corrected returns of n assets at time t such that $E(a_t) = 0$ and $Cov(a_t) = H_t$, μ_t is $n \times 1$ vector of the expected value of the conditional y_t , H_t is $n \times n$ matrix of conditional variances of y_t at time t , R_t is $n \times n$ conditional correlation matrix of a_t at time t, ε_t is $n \times 1$ vector of iid errors such that $E(\varepsilon_t) = 0$ and $E(\varepsilon_t \varepsilon_t^T) = I$, Q_t is $n \times n$ covariance matrix with time-varying and D_t is $n \times n$ diagonal matrix of conditional standard deviations of a_t at time t.

From univariate GARCH models, we took the elements in the diagonal matrix D_t .

$$D_t = \begin{bmatrix} \sqrt{h_{1t}} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{h_{2t}} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{h_{3t}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{h_{nt}} \end{bmatrix}$$

and h_{it} is given by :

$$h_{it} = a_{i0} + \sum_{q=1}^{Q_i} a_{iq} a_{i,t-q}^2 + \sum_{p=1}^{P_i} \beta_{ip} h_{i,t-p}$$

So, the correlation matrix R_t is symmetric :

$$R_t = \begin{bmatrix} 1 & \rho_{12,t} & \rho_{13,t} & \dots & \rho_{1k,t} \\ \rho_{12,t} & 1 & \rho_{23,t} & \dots & \rho_{2k,t} \\ \rho_{13,t} & \rho_{23,t} & 1 & \dots & \rho_{3k,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{1k,t} & \rho_{2k,t} & \rho_{3k,t} & \dots & 1 \end{bmatrix}$$

R_t has two requirements for determining its form :

1. The covariance matrix H_t has to be a positive definite so the conditional correlation matrix R_t has to be positive definite and the conditional standard deviations of a_t , D_t , has to be positive define since all the diagonal elements are positive.
2. All the elements of the R_t must be equal to or less than one by definition.

Thus, the dynamic conditional correlation matrix R_t can be written :

$$R_t = Q_t^{*-1} Q_t Q_t^{*-1}$$

$$Q_t = (1 - a - b)\overline{Q} + a\varepsilon_{t-1}\varepsilon_{t-1}^T + bQ_{t-1}$$

where :

- The unconditional covariance matrix of the standardized error is $\overline{Q} = Cov(\varepsilon_t \varepsilon_t^T) = E(\varepsilon_t \varepsilon_t^T)$ and this can be estimated as $\overline{Q} = 1/T \sum_{t=1}^{T} \varepsilon_t \varepsilon_t^T$
- a and b are parameters to be estimated and these must satisfy $a \geq 0$, $b \geq 0$ and $a + b < 1$

The Q_t^* is a diagonal matrix as follows :

$$Q_t^* = \begin{bmatrix} \sqrt{q_{11,t}} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{q_{22,t}} & 0 & \dots & 0 \\ 0 & 0 & \sigma_{3,t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{q_{nn,t}} \end{bmatrix}$$

and $|\rho_{ij}| = |\frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}}| \leq 1$ must apply for the second requirement.

If we have DCC(M,N)-GERCH model then the dynamic correlation structure is given that :

$$Q_t = (1 - \sum_{m=1}^{M} \alpha_m - \sum_{n=1}^{N} b_n)\overline{Q} + \sum_{m=1}^{M} \alpha_m \varepsilon_{t-1} \varepsilon_{t-1}^T + \sum_{n=1}^{N} b_n Q_{t-1}$$

6.6 Full-factor M-GARCH model

The multivariate full factor GARCH model (FFMG) (Vrontos, I.D., Dellaportas, P. and Politis, D.N., 2003) is defined as :

$$y_t = \mu + \varepsilon_t$$
$$\varepsilon_t = W X_t$$

$$X_t | \Phi_{t-1} \sim N_n(0, \Sigma_t)$$

where :

- μ is a $N \times 1$ vector of constants
- ε_t is a $N \times 1$ innovation vector and this has a linear combination of the factors $x_{i,t}$
- W is a $N \times N$ parameter matrix
- Φ_{t-1} is a set of information at time t-1
- X_t is a $N \times 1$ vector of factors and this consist of elements $x_{i,t}$ for $i = 1,....,N$
- Σ_t is a $N \times N$ variance covariance matrix and this has diagonal elements

The variance covariance matrix is given by :

$$\Sigma_t = \begin{bmatrix} \sigma^2_{1,t} & 0 & 0 & \dots & 0 \\ 0 & \sigma^2_{2,t} & 0 & \dots & 0 \\ 0 & 0 & \sigma^2_{3,t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma^2_{N,t} \end{bmatrix}$$

and we calculate the $\sigma^2_{i,t}$ with the following relationship :

$$\sigma^2_{i,t} = a_i + b_i x^2_{i,t-1} + f_i \sigma^2_{i,t-1} \quad \text{with} \quad i = 1,.....,N \text{ and } \quad t = 1,.....T$$

The $\sigma^2_{i,t}$ is the variance of the i factor and it must apply $a_i > 0$, $b_i \geq 0$ and $f_i \geq 0$. GARCH(1,1) processes follow the factors $x_{i,t}$.

Even if the vector ε_t follows a conditional multivariate normal distribution, that is $\varepsilon_t \,|\, \Phi_{t-1} \sim N(0,H_t)$, then we find the conditional covariance H_t of the asset returns at time t as follows :

$$H_t = W\Sigma_t W' = W\Sigma_t^{1/2}\Sigma_t^{1/2}W' = (W\Sigma_t^{1/2})(W\Sigma_t^{1/2})' = ZZ'$$

with

$$\Sigma_t^{1/2} = \begin{bmatrix} \sigma_{1,t} & 0 & 0 & \dots & 0 \\ 0 & \sigma_{2,t} & 0 & \dots & 0 \\ 0 & 0 & \sigma_{3,t} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_{N,t} \end{bmatrix}$$

The W is a below triangular matrix, that is, it is valid that $w_{ij} = 0$ for $j > i$ and $w_{ii} > 0$ for $i = 1,....,N$. But if we want to diminish the number of parameters of the model, then we will use a natural restriction which is $w_{ii} = 1$ for $i = 1,....,N$. This constraints have been used by Aguilar and West (2000), Geweke and Zhou (1996) and Chib et al. (1998). The conditional covariance matrix H_t is defined according to the constraint $w_{ii} = 1$ as follows :

$$H_t = W\Sigma_t W' = \begin{bmatrix} h_{11,t} & h_{12,t} & h_{13,t} & \dots & h_{1N,t} \\ h_{21,t} & h_{22,t} & h_{23,t} & \dots & h_{2N,t} \\ h_{31,t} & h_{32,t} & h_{33,t} & \dots & h_{3N,t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{N1,t} & h_{N2,t} & h_{N3,t} & \dots & h_{NN,t} \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_{1,t}^2 & w_{21}\sigma_{1,t}^2 & w_{31}\sigma_{1,t}^2 & \dots & w_{N1}\sigma_{1,t}^2 \\ w_{21}\sigma_{1,t}^2 & \sum_{i=1}^{2} w_{2i}^2\sigma_{i,t}^2 & \sum_{i=1}^{2} w_{2i}w_{3i}\sigma_{i,t}^2 & \dots & \sum_{i=1}^{2} w_{2i}w_{Ni}\sigma_{i,t}^2 \\ w_{3i}^2\sigma_{1,t}^2 & \sum_{i=1}^{2} w_{3i}w_{2i}\sigma_{i,t}^2 & \sum_{i=1}^{3} w_{3i}^2\sigma_{i,t}^2 & \dots & \sum_{i=1}^{3} w_{3i}w_{Ni}\sigma_{i,t}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{N1}\sigma_{1,t}^2 & \sum_{i=1}^{2} w_{Ni}w_{2i}\sigma_{i,t}^2 & \sum_{i=1}^{3} w_{Ni}w_{3i}\sigma_{i,t}^2 & \dots & \sum_{i=1}^{N} w_{Ni}^2\sigma_{i,t}^2 \end{bmatrix}$$

If the factor variances $\sigma_{i,t}^2$ are well defined for $i = 1,....,N$, then the variance covariance matrix H_t will be positive definite such as we come to the attention of the construction of the model. Moreover, the $x_{i,t}$ are not parameters to be estimated and these have zero idiosyncratic variances. Indeed, these are given by $X_t = W^{-1}\varepsilon_t$. In the event that a model has $N = 1$, we have GARCH(1,1) .

6.7 Regime switching dynamic correlation model

We know that the variance and covariance of financial time series are time-varying. Many models face additional problems when we write a multivariate model of volatility. The Regime Switching Dynamic Correlation model (RSDC) is a new multivariate volatility model and it has better performance at capturing correlation asymmetries. A special case of the RSDC model with only one regime is the CCC model (Pelletier, D., 2006). The Regime Switching Dynamic Correlation model is defined by :

$$y_t = \mu + \varepsilon_t$$
$$\varepsilon_t \,|\, \Phi_{t-1} \sim N_n(0, H_t)$$

The covariance matrix H_t can be decomposed into :

$$H_t = \Sigma_t \Gamma_t \Sigma_t$$

where :

- Σ_t is a diagonal matrix which contains the standard deviations $\sigma_{k,t}$ and is time-varying at time t for $k = 1,....,K$ where K: number of time series
- Γ_t contains the correlations which is time-varying at time t

Moreover, the variance $\sigma_{k,t}^2$ calculates by :

$$\sigma_{k,t}^2 = a_k + b_k x_{k,t-1}^2 + f_k \sigma_{k,t-1}^2 \quad \text{with} \quad k = 1,.....,K \text{ and } \quad t = 1,.....T$$

However, the Γ_t follows a Markov Chain and this has different values for different regimes. According to a Markov chain process, the RSDC generates dynamic correlations because of transitioning between regimes of different correlation levels. The switch from one regime to another is determined by transition probabilities and these are time-varying.

The correlation matrix modeled Γ_t with based on a dynamic framework as follows :

$$\Gamma_t = \sum_{n=1}^{N} 1_{(\Delta_t = n)} \Gamma_n$$

where :

- Δ_t is an unserved Markov chain process independent of ε_t taking also N possible values ($\Delta_t = 1,2,....,N$)
- 1 is an indicator function
- Γ_n is $K \times K$ correlation matrices with $\Gamma_n \neq \Gamma'_n$ for $n \neq n'$, off-diagonal elements are -1 and 1

The probability law governing the Markov chain process Δ_t is defined as Π_t. Also, the $\pi_t^{i,j}$ is the probability of going from regime i in period t-1 to regime j in period t and it is found with $\pi_t^{i,j} = P(\Delta_t = j \mid \Delta_{t-1} = i)$ [8]. Moreover, the π^n denote the limiting probability of being in regime n.

The Γ_n and Γ_t are correlation matrix. The Γ_n contains the off-diagonal elements which are between -1 and 1 and the diagonal elements are 1. We must impose that Γ_n will be a correlation matrix and this can become with the Choleski decomposition. Thus, we have $\Gamma_n = P_n P'_n$. Provided that the P_n is a lower triangular matrix and impose constraints on P_n for the purpose of getting ones on the diagonal. So, these constraints will have off diagonal elements between [-1,1]. The Γ_n comes to be with the Choleski decomposition as follows for trivariate example :

$$\Gamma = \begin{bmatrix} p_{1,1}^2 & p_{1,1}p_{2,1} & p_{1,1}p_{3,1} \\ p_{1,1}p_{2,1} & p_{2,1}^2 p_{2,2}^2 & p_{2,1}p_{3,1} + p_{2,2}p_{3,2} \\ p_{1,1}p_{3,1} & p_{2,1}p_{3,1} + p_{2,2}p_{3,2} & p_{3,1}^2 + p_{3,2}^2 + p_{3,3}^2 \end{bmatrix}$$

The elements on the diagonal P_n are positive and the constraints $\Gamma_{jj} = 1$ becomes as follows:

$$p_{j,j} = \sqrt{1 - \sum_{i=1}^{j-1} p_{j,i}^2} \qquad \text{for} \quad j = 1,......,K$$

where if we have $j = 1$, then the sum is zero.The estimation of the RSDC model will be complicated by the high number of parameters with each Γ_n, so we can use the EM algorithm which will not complicate the estimation from increasing the number of time series.

Chapter 7

Application to real data

7.1 Introduction

In this Chapter, we use an example of real data to construct an optimal portfolio and applied some models in order to predict the returns of financial data and to estimate the variability and participation of their returns. The analysis is based on equity funds which are a type of hedge funds or private investment funds and these invest principally in stocks. We differently call them and as stock funds. A market capitalization determines the size of an equity fund and the investment style is also used to categorize equity hedge funds which reflected in the find's stock holding. These funds are categorized whether they are domestic or international.

We analyze below a real dataset of returns equity funds from the United States. So, in order to construct and create multivariate multi-factor models, we employ thirty-one years of monthly data, for the period 01/01/1987 to 01/01/2018. We specifically have dependent variables (monthly returns) which are twenty and are the following (figure 7.1) :

- BMCAX US Equity (BlackRock Advantage Large Cap Growth Fund)
- FDCAX US Equity (Fidelity Capital Appreciation Fund)
- FCNTX US Equity (Fidelity Contrafund)
- FEQIX US Equity (Fidelity Equity-Income Fund Inc)
- FGRIX US Equity (Fidelity Growth & Income Portfolio)
- FDFFX US Equity (Fidelity Independence Fund)
- FMAGX US Equity (Fidelity Magellan Fund)

- FDVLX US Equity (Fidelity Value Fund)
- FTRNX US Equity (Fidelity Trend Fund)
- FKGRX US Equity (Franklin Growth Fund)
- FKDNX US Equity (Franklin DynaTech Fund)
- PRNHX US Equity (T Rowe Price New Horizons Fund Inc)
- PRGIX US Equity (T Rowe Price Growth & Income Fund Inc)
- SHRAX US Equity (ClearBridge Aggressive Growth Fund)
- LMASX US Equity (ClearBridge Small Cap Fund)
- CHTRX US Equity (Invesco Charter Fund)
- OPOCX US Equity (Invesco Oppenheimer Discovery Fund)
- QUASX US Equity (AB Small Cap Growth Portfolio)
- CABDX US Equity (AB Relative Value Fund)
- CHCLX US Equity (AB Discovery Growth Fund Inc)

In addition to the returns, we also have the factors that affect them. Thus, we have the following eight factors which are calculated following the approach by Fama and French (2015):

- Mkt-RF (market risk premium) is the market factor which is calculated as the value weighted average of the returns of all stocks in the region minus the monthly returns on one-month U.S.
- SMB (small minus big) is the average return on the small stock portfolios minus the average return on the big stock portfolios.
- HML (high minus low) is the average return between the returns on diversified portfolios of high book-to-market[1] (B/M) and low book-to-market[1] (B/M) stocks.
- RMW (robust minus weak) is the difference between returns on firms with robust profitability and weak profitability.
- MOM is the momentum factor
- BAB (betting-against-beta) is long leveraged low-beta assets and short high-beta assets.
- CAR is an additional factor

[1] book-to-market ratio is used to find the value of a company by comparing its book value to its market value $Book-to-Market\ Ratio = \frac{Common\ Shareholders\ Equity}{Market\ Cap}$

We present an optimal minimum variance portfolios of the form:

$$\min_{w} \frac{1}{2} V(R_{p,t}) = min\{\frac{1}{2} w' \Sigma_t w\}$$

$$s.t. \sum_{i=1}^{n} w_i = 1$$

and mean-variance portfolios of the form:

$$\min_{w} \frac{1}{2} V(R_{p,t}) = min\{\frac{1}{2} w' \Sigma_t w\}$$

$$w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \quad \text{and} \quad E(R_{p,t}) \geq r_{Target}$$

The target return will be 0.005 on a monthly basis. We will construct portfolios for the out-of-sample period 1/2013 - 01/2018 (60 months). The estimation of the mean vector and the covariance matrix should be estimated using the following methods: Sample estimate of mean and covariance matrix, based on the single model, based on multivariate multiple regression models, based on the Constant Conditional Correlation for the variance-covariance matrix and based on the multivariate heteroskedastic model. As we have mentioned in previous chapters, the portfolio construction problem can be considered as a selection problem of assets, and a problem determining the portfolio weights. The investors want to construct portfolios that minimize the portfolio risk (portfolio standard deviation) for a given target return or to maximize the expected portfolio return given a specific portfolio risk. As it was previously stated in this chapter the target return is 0.005 and thus, we want to minimize the portfolio risk.

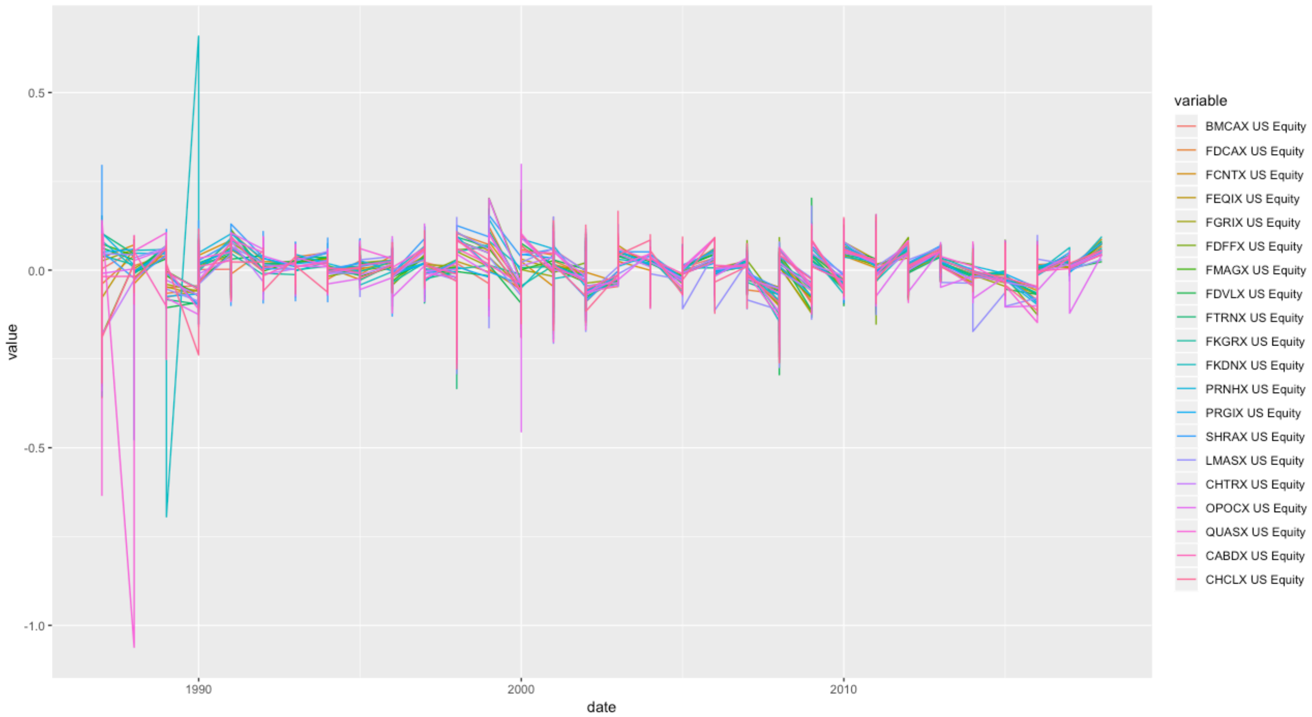


FIGURE 7.1 : Plot of the returns from the funds

7.2 Sample Estimate

The first attempt is to construct minimum variance and mean variance portfolios, estimating the mean vector and the covariance matrix by using the method of sample estimate of mean vector and covariance matrix. We remind that the out of sample period is 60 months and we have 20 funds.

The weights from the method mean-variance of each funds for the first and the last month of the 60 out of sample months are represented in Table 7.1.

Funds /Weights	1st month of 60	60th month of 60
BMCAX US Equity	0	0
FDCAX US Equity	0	0
FCNTX US Equity	0.53614778	0.2501056
FEQIX US Equity	0	0
FGRIX US Equity	0	0
FDFFX US Equity	0	0
FMAGX US Equity	0	0
FDVLX US Equity	0	0
FTRNX US Equity	0	0
FKGRX US Equity	0.44449711	0.5818623
FKDNX US Equity	0	0
PRNHX US Equity	0	0
PRGIX US Equity	0	0
SHRAX US Equity	0	0
LMASX US Equity	0	0
CHTRX US Equity	0	0
OPOCX US Equity	0	0
QUASX US Equity	0	0
CABDX US Equity	0.01935511	0.1680321
EVSEX US Equity	0	0

TABLE 7.1 : Weights from mean-variance portfolios for the 20 funds the first and the last month of out of sample period.

The weights from the method minimum variance of each funds for the first and the last month of the 60 out of sample months are represented in Table 7.2.

Funds /Weights	1st month of 60	60th month of 60
BMCAX US Equity	0	0
FDCAX US Equity	0	0
FCNTX US Equity	0.2104963	0.2105332
FEQIX US Equity	0	0
FGRIX US Equity	0	0
FDFFX US Equity	0	0
FMAGX US Equity	0	0
FDVLX US Equity	0	0
FTRNX US Equity	0	0
FKGRX US Equity	0.6035629	0.5987569
FKDNX US Equity	0	
PRNHX US Equity	0	0
PRGIX US Equity	0	0
SHRAX US Equity	0	0
LMASX US Equity	0	0
CHTRX US Equity	0	0
OPOCX US Equity	0	0
QUASX US Equity	0	0
CABDX US Equity	0.1859408	0.1907099
CHCLX US Equity	0	0

TABLE 7.2 : Weights from minimum variance portfolios for the 20 funds the first and the last month of out of sample period.

In the 1st month the fund FCNTX US Equity had the highest value in weights for the mean-variance portfolios, but the FKGRX US Equity had the highest value in weights for the minimum variance portfolios in the 60th month. We can see that the fund FKGRX US Equity had the highest values in weights not only for the mean-variance but as well for the minimum variance portfolios. It is observed that there are many funds which have zero weight values.

In Table 7.3, we present the out of sample mean return, the portfolio standard deviation (volatility), the cumulative returns and the conditional Sharpe Ratio.

	Mean Return	**Volatility**	**Cumulative Return**	**Conditional Sharpe Ratio**
Minimum Variance	0.01192118	0.03874101	0.7152706	0.3079726
Mean-Variance	0.01217711	0.03915046	0.7306267	0.3105148

TABLE 7.3 : Mean return, Volatility, Cumulative return and Conditional Sharpe Ratio for the minimum portfolio and the mean-variance portfolio.

The Mean Return, Volatility, Cumulative Return of the mean-variance portfolio is higher than the corresponding minimum variance portfolio and the Conditional Sharpe Ratio, too.

In Figure 7.2, the cumulative returns of each portfolio are represented for the out of sample period. We conclude that until the 10^{th} month of the out of sample period the cumulative returns of minimum variance and mean-variance portfolio essentially coincide. From then on until the 60^{th} month of the out of sample period they are in very good agreement.After, these seem to identical the 47^{th} month until the 51^{st} month. Finally, these seem that until the 60^{th} month of the out of sample period the cumulative returns of mean-variance portfolio is higher than the cumulative returns of minimum variance portfolio again.

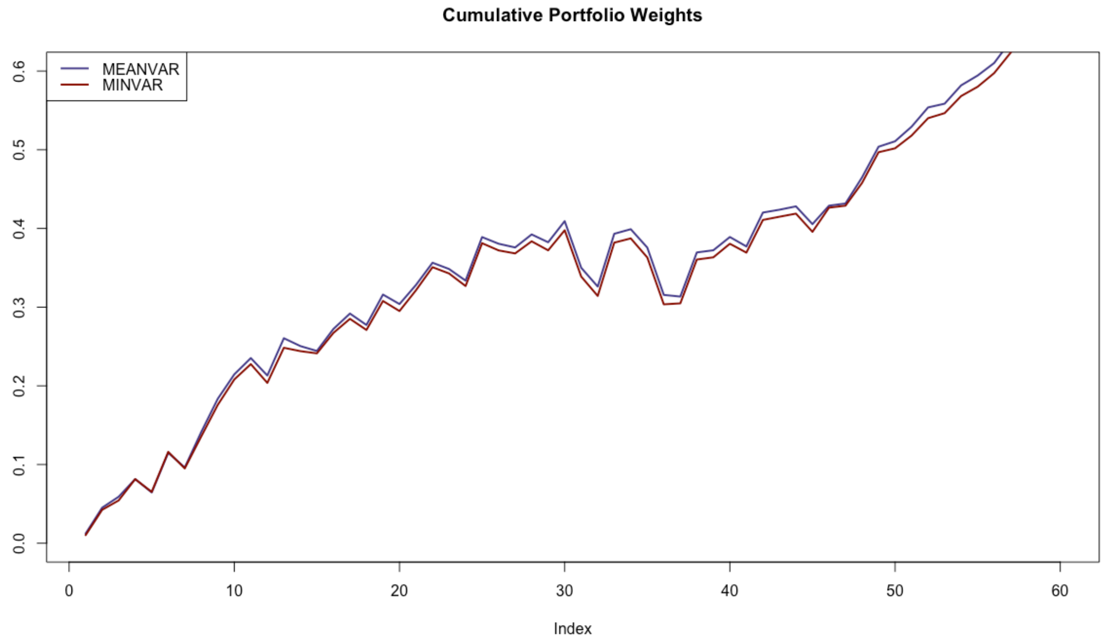


FIGURE 7.2 : Plot of Cumulative returns for the mean-variance portfolio and for the minimum variance portfolio.

7.3 Based on Single Index Model

In this section, we construct mean-variance and minimum variance portfolios, estimating the mean vector and the covariance matrix based on the Single Index Model. As we have mentioned in previous chapter, the Single Index Model is given by :

$$R_{i,t} = a_i + \beta_i \cdot R_{M,t} + \varepsilon_{i,t}, \varepsilon_{i,t} \sim N(0,\sigma^2_{i,\varepsilon}), \qquad i = 1,.....,20, \quad t = 1,.....,60$$

The weights from the method mean-variance of each fund for the first and the last month of the 60 out of sample months are represented in Table 7.4.

Funds /Weights	**1^{st} month of 60**	**60^{th} month of 60**
BMCAX US Equity	0	0.05599
FDCAX US Equity	0	0.04833
FCNTX US Equity	0.53463	0.08907
FEQIX US Equity	0	0.05967
FGRIX US Equity	0	0.06108
FDFFX US Equity	0	0.03397
FMAGX US Equity	0	0.04991
FDVLX US Equity	0	0.05062
FTRNX US Equity	0	0.04669
FKGRX US Equity	0.41053	0.07572
FKDNX US Equity	0.02372	0.05837
PRNHX US Equity	0	0.03889
PRGIX US Equity	0	0.07210
SHRAX US Equity	0	0.04020
LMASX US Equity	0	0.02162
CHTRX US Equity	0	0.05782
OPOCX US Equity	0	0.01898
QUASX US Equity	0	0.03047
CABDX US Equity	0.03112	0.06024
EVSEX US Equity	0	0.03025

TABLE 7.4 : Weights from mean-variance portfolios for the 20 funds the first and the last month of out of sample period.

The weights from the method minimum variance of each asset for the first and the last month of the 60 out of sample months are represented in Table 7.5.

Funds /Weights	1st month of 60	60th month of 60
BMCAX US Equity	0.08414	0.05465
FDCAX US Equity	0	0.04570
FCNTX US Equity	0	0.07703
FEQIX US Equity	0.04738	0.06506
FGRIX US Equity	0	0.06754
FDFFX US Equity	0	0.03259
FMAGX US Equity	0	0.05142
FDVLX US Equity	0	0.04712
FTRNX US Equity	0	0.04623
FKGRX US Equity	0.54299	0.07587
FKDNX US Equity	0.01051	0.05474
PRNHX US Equity	0	0.03261
PRGIX US Equity	0.10240	0.07618
SHRAX US Equity	0	0.03554
LMASX US Equity	0	0.02772
CHTRX US Equity	0	0.06333
OPOCX US Equity	0	0.02420
QUASX US Equity	0	0.03005
CABDX US Equity	0.21259	0.06182
CHCLX US Equity	0	0.03061

TABLE 7.5 : Weights from minimum variance portfolios for the 20 funds the first and the last month of out of sample period.

For the mean - variance portfolio the 1st and the 60th month the fund FCNTX US Equity had the highest values in weights for both the mean-variance and only in the 60th month it had the highest values in weights for the minimum variance portfolio. However, the fund PRGIX US Equity is presented with the highest value in the 1st month.

In Table 7.6, we present the out of sample mean return, the portfolio standard deviation (volatility), the cumulative returns and the conditional Sharpe Ratio.

	Mean Return	Volatility	Cumulative Return	Conditional Sharpe Ratio
Minimum Vaiance	0.0103	0.0113	0.6154	0.9371
Mean-Variance	0.0112	0.0140	0.6711	0.8169

TABLE 7.6 : Mean return, Volatility, Cumulative return and Conditional Sharpe Ratio for the minimum portfolio and the mean-variance portfolio.

The Mean Return, Volatility and Cumulative Return of mean-variance portfolio seem to be higher than the corresponding of minimum variance portfolio. Nevertheless, the Conditional Sharpe Ratio is presented to be highest for the minimum variance portfolio.

In figure 7.3, the cumulative returns of each portfolio are represented for the out of sample period. We conclude that through the months of out of sample period the cumulative returns are higher for the mean-variance portfolio than the minimum variance portfolio. Except of the 1st month until the 3rd month when the cumulative return of the minimum variance portfolio is the same the cumulative return of the mean-variance portfolio.

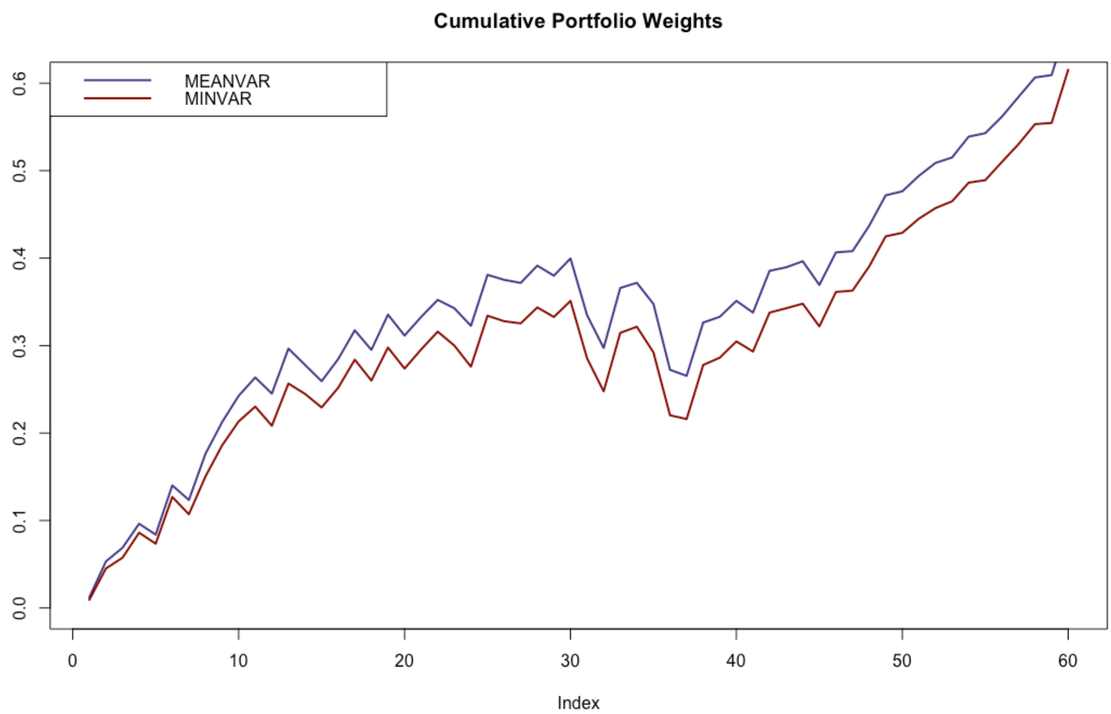


FIGURE 7.3 : Plot of Cumulative returns for the mean-variance portfolio and for the minimum variance portfolio with Single Index Model

7.4 Constant Conditional Correlation Model

Now, we construct mean-variance and minimum variance portfolios, estimating the mean vector and the covariance matrix considering a Constant Conditional Correlation for the variance covariance matrix of the form :

$$R_t = X_t\Gamma + E_t \sim N(0,H_t), \quad t = 1,2,....,T \quad \text{with} \quad H_t = D_t R D_t (= \rho_{ij}\sqrt{\sigma_{i,t}^2\sigma_{j,t}^2})$$

Funds /Weights	1st month of 60	60th month of 60
BMCAX US Equity	0.21945	0
FDCAX US Equity	0.11972	0
FCNTX US Equity	0	0
FEQIX US Equity	0	0
FGRIX US Equity	0	0
FDFFX US Equity	0	0
FMAGX US Equity	0	0
FDVLX US Equity	0	0.07637
FTRNX US Equity	0	0
FKGRX US Equity	0.63616	0
FKDNX US Equity	0	0
PRNHX US Equity	0	0
PRGIX US Equity	0	0
SHRAX US Equity	0	0
LMASX US Equity	0	0
CHTRX US Equity	0	0.92363
OPOCX US Equity	0	0
QUASX US Equity	0	0
CABDX US Equity	0.02467	0
EVSEX US Equity	0	0

TABLE 7.7 : Weights from mean-variance portfolios for the 20 funds the first and the last month of out of sample period.

Funds /Weights	1st month of 60	60th month of 60
BMCAX US Equity	0	0
FDCAX US Equity	0.17081	0
FCNTX US Equity	0	0
FEQIX US Equity	0	0
FGRIX US Equity	0	0
FDFFX US Equity	0	0
FMAGX US Equity	0	0
FDVLX US Equity	0	0.07637
FTRNX US Equity	0	0
FKGRX US Equity	0.66181	0
FKDNX US Equity	0	0
PRNHX US Equity	0	0
PRGIX US Equity	0	0
SHRAX US Equity	0	0
LMASX US Equity	0	0
CHTRX US Equity	0	0.92363
OPOCX US Equity	0	0
QUASX US Equity	0	0
CABDX US Equity	0.16738	0
CHCLX US Equity	0	0

TABLE 7.8 : Weights from minimum variance portfolios for the 20 funds the first and the last month of out of sample period.

The weights from the method mean-variance of each asset for the first and the last month of the 60 out of sample months are represented in Table 7.7.

The weights from the method minimum variance of each asset for the first and the last month of the 60 out of sample months are represented in Table 7.8.

It is obvious that the weights of each asset are the same for both mean-variance and minimum variance portfolios 60th month only. In the 1st and in the 60th month, the FKGRX US Equity and CHTRX US Equity had the highest weights, respectively

In Table 7.9, we present the out of sample mean return, the portfolio standard deviation (volatility), the cumulative returns and the conditional Sharpe Ratio.

	Mean Return	Volatility	Cumulative Return	Conditional Sharpe Ratio
Minimum Vaiance	0.0089	0.0275	0.5370	0.3557
Mean-Variance	0.0090	0.0275	0.5376	0.3554

TABLE 7.9 : Mean return, Volatility, Cumulative return and Conditional Sharpe Ratio for the minimum portfolio and the mean-variance portfolio.

The mean return, Cumulative return of mean variance portfolio is higher than the mean return of minimum variance portfolio and the Sharpe ratio, too.

In figure 7.4, the cumulative returns of each portfolio are represented for the out of sample period. We conclude that all the months of the out of sample period the cumulative returns are same for the mean-variance portfolio than the minimum variance portfolio. From then on, the cumulative return of the mean-variance portfolio is higher than the cumulative return of the minimum variance portfolio.

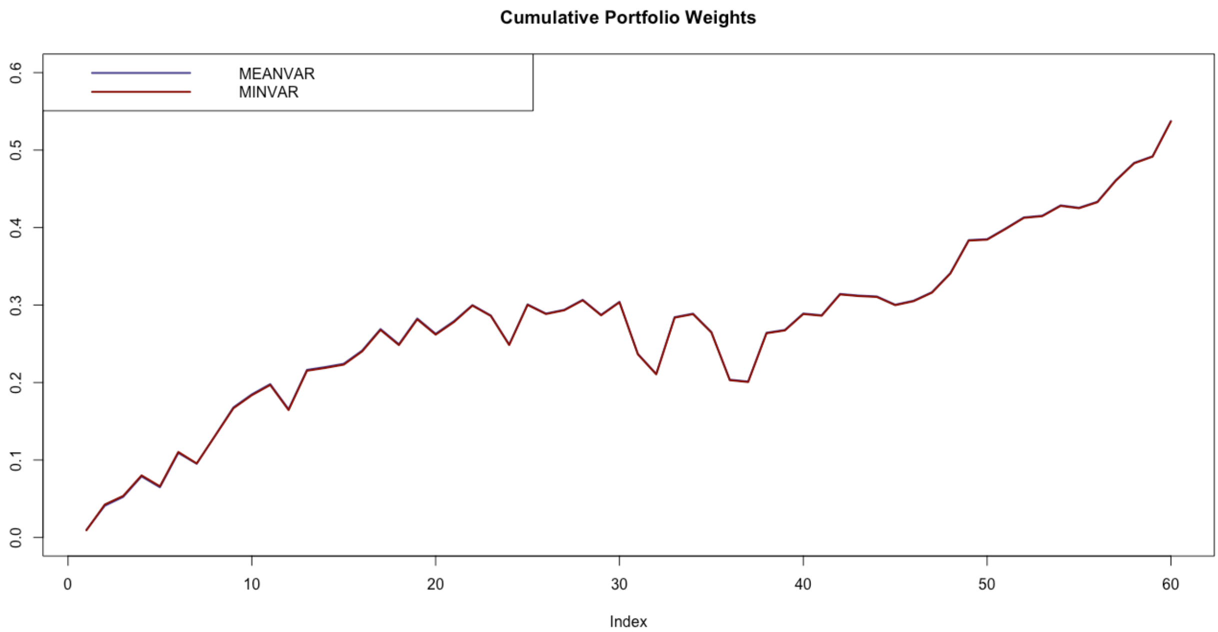


FIGURE 7.4 : Plot of Cumulative returns for the mean-variance portfolio and for the minimum variance portfolio with CCC Model.

7.5 Multivariate Multiple Regression Model

We construct mean-variance and minimum variance portfolios, estimating the mean vector and the covariance matrix based on multivariate multiple regression models :

$$R_t = X_t \Gamma' + E_t, \qquad E_t \sim N(0, \Sigma), \qquad t = 1, \ldots\ldots, T$$

The weights from the method mean-variance of each fund for the first and the last month of the 60 out of sample months are represented in table 7.9.

Funds /Weights	1st month of 60	60th month of 60
BMCAX US Equity	0	0
FDCAX US Equity	0	0
FCNTX US Equity	0.53614778	0.4218525
FEQIX US Equity	0	0
FGRIX US Equity	0	0
FDFFX US Equity	0	0
FMAGX US Equity	0	0
FDVLX US Equity	0	0
FTRNX US Equity	0	0
FKGRX US Equity	0.44449711	0.2909520
FKDNX US Equity	0	0
PRNHX US Equity	0	0
PRGIX US Equity	0	0.2871955
SHRAX US Equity	0	0
LMASX US Equity	0	0
CHTRX US Equity	0	0
OPOCX US Equity	0	0
QUASX US Equity	0	0
CABDX US Equity	0.01935511	0
EVSEX US Equity	0	0

TABLE 7.9 : Weights from mean-variance portfolios for the 20 funds the first and the last month of out of sample period.

The weights from the method minimum variance of each fund for the first and last month of the 60 out of sample months appear in table 7.10.

Funds /Weights	1st month of 60	60th month of 60
BMCAX US Equity	0	0
FDCAX US Equity	0	0
FCNTX US Equity	0.1962740	0.4218525
FEQIX US Equity	0	0
FGRIX US Equity	0	0
FDFFX US Equity	0	0
FMAGX US Equity	0	0
FDVLX US Equity	0	0
FTRNX US Equity	0	0
FKGRX US Equity	0.6105099	0.2909520
FKDNX US Equity	0	0
PRNHX US Equity	0	0
PRGIX US Equity	0	0.2871955
SHRAX US Equity	0	0
LMASX US Equity	0	0
CHTRX US Equity	0	0
OPOCX US Equity	0	0
QUASX US Equity	0	0
CABDX US Equity	0.1932161	0
CHCLX US Equity	0	0

TABLE 7.10 : Weights from minimum variance portfolios for the 20 funds the first and the last month of out of sample period.

We observe that the fund FCNTX US Equity is highest values of weights in relation to the others funds for mean-variance portfolios in the 1st and in the 60th month. On the other hand, the fund FCNTX US Equity is highest value of weight only the 60th month and the FKGRX US Equity is highest value in the 1st month.

We found the Mean Returns, the Cumulative Returns and the Conditional Sharpe Ratio for the out of sample period 1/2013 - 01/2018 and we can see these in the table 7.11.

	Mean Return	Volatility	Cumulative Return	Conditional Sharpe Ratio
Minimum Vaiance	0.01208	0.03651	0.72464	0.33234
Mean-Variance	0.01226	0.03681	0.73541	0.33242

TABLE 7.11 : Mean return, Volatility, Cumulative return and Conditional Sharpe Ratio for the minimum portfolio and the mean-variance portfolio.

The Mean Returns, the Cumulative Returns and the conditional Sharpe Ratio of mean-variance portfolio is higher than the Mean Returns, the Cumulative Returns and the Conditional Sharpe Ratio of minimum variance portfolio.

In figure 7.5, the cumulative returns of each portfolio are represented for the out of sample period. We conclude that until the 9th month of the out of sample period the cumulative return of minimum variance portfolio is equal to the cumulative return of the mean-variance portfolio. From then until the 60th month of the out of sample period the cumulative return of the mean-variance portfolio is higher than the cumulative return of

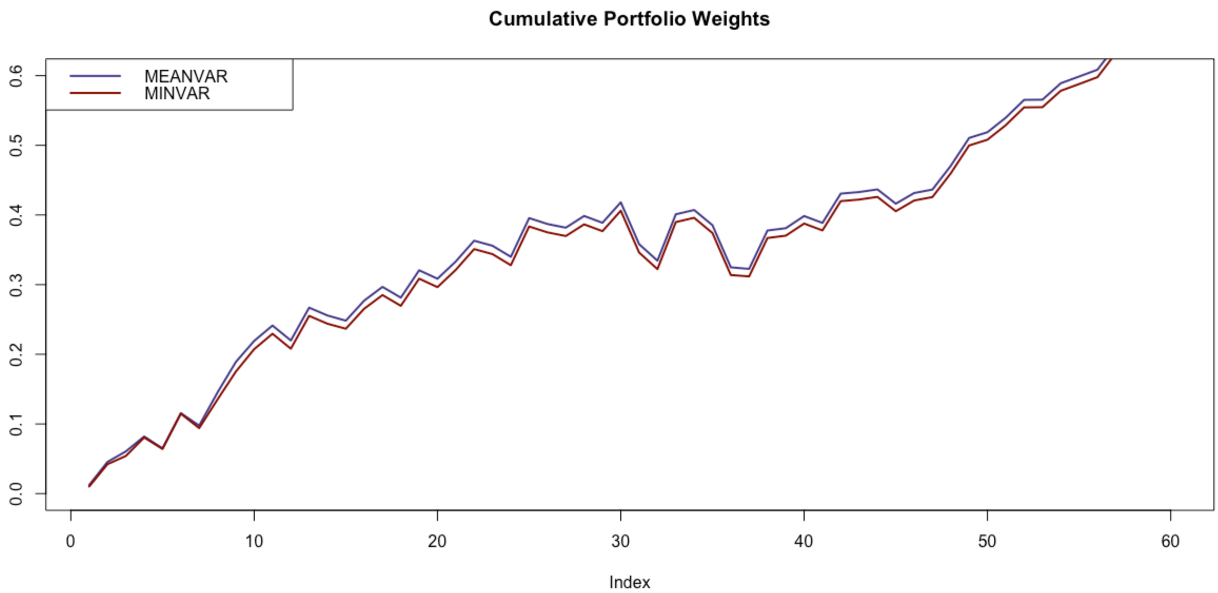


FIGURE 7.5 : Plot of Cumulative returns for the mean-variance portfolio and for the minimum variance portfolio with Multivariate Multiple Model.

minimum variance portfolio.

Chapter 8

Concluding Remarks

An economist-analyst to reach the best decision must follow a multidimensional and analytical process. The return is the main subject of consideration and the risk, that is volatility, the main barrier to profit. In order to be able to measure with relative precision the interaction between these two elements, this relationship is necessary to identify and model. The financial data are a dynamic investment both the variance and covariance are time-varying. The time series of returns demonstrate volatility clustering and high kurtosis. In this thesis, we concentrated on time varying of the variances and covariances of returns, the risk measurement and we focused on building mutual fund portfolio construction by creating models. We presented some models and thus, we saw different methods of forecasting variances and covariances. Moreover, we constructed optimal mutual fund portfolios and measure tail-risk.

We find that a single index model, SIM, reduces portfolio risk and improves the out of sample risk adjusted realized returns. We also find that the CSR of the portfolio constructed with the SIM model is the higher among alternative models. This suggests that the SIM covariance model represents a more accurate tool for tail-risk measurement. We constructed models with minimum variance and mean-variance, thus we saw how the minimum variance from the SIM model is higher than mean-variance from the SIM model respectively. Furthermore, we observed that the returns of FCNTX US Equity greatly affect because of this seem to have the highest weight in most models both minimum variance and mean-variance.

Bibliography

[1] Elton, E.J., Gruber, M.J., Brown, S.J. and Goetzmann W.N. (2014). Modern Portfolio Theory and Investment Analysis, 9th edition, Wiley

[2] Fama, E. and French, . (1993). Common risk factors in the returns of stocks and bonds. Journal of Financial Economics, 33, 3-56

[3] Woohwan Kim, Young Min Kim, Tae-Hwan Kim, Seungbeom Bang, Multi-dimensional portfolio risk and its diversification: A note, Global Finance Journal, 147-156

[4] Taras Bodnar, Nestor Parolya, Wolfgang Schmid (2018), Estimation of the global minimum variance portfolio in high dimensions, European Journal of Operational Research, 371-390

[5] Hany Fahmy (2019), Mean-variance-time: An extension of Markowitz's mean-variance portfolio theory, Journal of Economics and Business

[6] Salvador Cruz Rambaud, José García Pérez, Miguel Angel Sánchez Granero, Juan Evangelista Trinidad Segovia (2005), Theory of portfolios: New considerations on classic models and the Capital Market Line, European Journal of Operational Research, 276-283

[7] Sharpe, W. F. (1964). Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. Journal of Finance, 19, 3, 425 42.

[8] Ross, S. A. (1976). The arbitrage theory of capital asset pricing, Journal of Economic Theory, 341 – 360.

[9] Blake, C. R., Elton, E. J., and Gruber, M. J. (1993). The performance of bond mutual funds. The Journal of Business, 66, 3, 371-403

[10] Lei Huang, Hui Jiang, Huixia Wang (2019), A novel partial-linear single-index model for time

series data, Computational Statistics & Data Analysis 110-122

[11] Jaqueson K. Galimberti (2019), An approximation of the distribution of learning estimates in macroeconomic models, Journal of Economic Dynamics and Control 29-43

[12] A. J. O'Malley, B. H. Neelon (2014), Latent Factor and Latent Class Models to Accommodate Heterogeneity, Using Structural Equation, Encyclopedia of Health Economics 131-140

[13] Diaa Noureldin, Neil Shephard, Kevin Sheppard (2014), Multivariate rotated ARCH models, Journal of Econometrics, 16-30

[14] Vrontos, I.D., P. Dellaportas, and Politis, D.N. (2003). Inference for some multivariate ARCH and GARCH models. Journal of Forecasting, 22, 427-446.

[15] Federico Poloni, Giacomo Sbrana (2014), Feasible generalized least squares estimation of multivariate GARCH(1, 1) models, Journal of Multivariate Analysis 151-159

[16] Farid Boussama, Florian Fuchs, Robert Stelzer (2011), Stationarity and geometric ergodicity of BEKK multivariate GARCH models, Stochastic Processes and their Applications, 2331-2360

[17] Begoña Fernández, Nelson Muriel (2009), Regular variation and related results for the multivariate GARCH(p,q) model with constant conditional correlations, Journal of Multivariate Analysis, 1538-1550

[18] Engle, R.F. (2002). Dynamic Conditional Correlation: A Simple Class of Multivariate GARCH Models. Journal of Business and Economic Statistics, 20, 339–350.

[19] Vrontos, I.D., Dellaportas, P. and Politis, D.N. (2003). A full-factor Multivariate GARCH model. Econometric Journal, 6, 312–334.

[20] Pelletier, D. (2006). Regime switching for dynamic correlations. Journal of Econometrics, 131, 445-73.