Portfolio Construction Using Parametric and Non-Parametric Methods

MSc in International Economics & Finance

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Abstract

The main purpose of this dissertation is to study the most important optimization models to mitigate financial risks. Especially, we want to construct optimal portfolios using the concepts of *first-order* stochastic dominance, *second-order* stochastic dominance and *third-order* stochastic dominance as well as the $CVaR$ approach, including three different investment tactics. The concept of Stochastic Dominance is theoretically appealing: if a return distribution A first-order, second-order, third-order stochastically dominates another distribution B, then all investors with some specific preferences will prefer A to B.

Portfolio optimization is a cornerstone of modern finance theory, as it is very attractive in the field of decision making under uncertainty. Financial crises, economic imbalances, algorithmic trading and highly volatile movements of asset prices in the recent times have raised high alarms on the management of financial risks. Inclusion of risk measures towards balancing optimal portfolios has become very crucial and equally critical. Formally, financial portfolio optimization adheres to a formal approach in making investment decisions, (1) for selection of investment portfolios containing the financial instruments, (2) to mitigate financial risks and ensure better preparedness for uncertainties, (3) to establish mathematical and computational methods on realistic constraints and (4) to provide stability across inter and intraday market fluctuations. Risk management has been recognized to play an increasingly important role in financial problems such as the international asset allocation, where widespread deregulation has entailed a substantial increase in asset price and currency volatility.

We start with an introduction of the mean-variance approach, as well as with a description of the different kinds of financial risks that are faced by investors and financial institutions. Moreover, we present the major risk measures used in portfolio optimization such as variance, mean-absolute deviation, Value at Risk, Conditional Value at Risk and the associated mathematical formulations of the optimization models. Next we focus on the Utility Theory and on how people make choices when faced with uncertainty, leading up to the development of the first, second and third stochastic dominance rules. Furthermore we define the key concepts and present the appropriate mathematical formulations in order to construct optimal portfolios based on the $CVaR$ optimization model, including three different investment tactics, as well as the FSD, SSD and TSD efficiency algorithms developed by Kuosmanen (2001,2004). More specifically in the empirical tests we consider investments in the US market. We want to construct several optimal

portfolios based on alternative strategies. We use data on monthly closing prices of S&P500, including a number of stocks obtained by Datastream covering the period from December 1999 to July 2016. We choose assets from different sectors and thus a total number of 30 assets are concerned in each portfolio. We conduct both static test(efficient frontier), considering the $CVaR$ optimization model, and dynamic tests in order to find the optimal portfolio weights (backtesting experiments over the last 120 months). Finally, we examine the statistical characteristics of the historical data set, we describe briefly the computational tests and we compare the statistical characteristics of the optimal portfolios. Moreover we employ four commonly used parametric performance measures in order to evaluate the performance of all the alternative competing strategies with respect to the market benchmark portfolio: the Sharpe ratio, the Sortino ratio, opportunity cost and portfolio turnover.

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Chapter 1

Introduction

1.1 Dissertation Theme and Motivation

Asset managers aim to select investment portfolios that yield the maximum possible return, while at the same time ensuring an acceptable level of risk exposure. Risk derives from potential losses in portfolio value due to possible reductions in the market value of financial assets resulting from changes in equity prices, interest rates, foreign exchange rates, etc. The theory of optimal portfolio selection was developed by Markowitz in the 1950's. His work formalized the diversification principles in portfolio selection and earned him the 1990 Nobel prize in economics. Since then, mathematical programming techniques have become essential tools in financial management, and thus are being increasingly applied in practice. The need to adopt sophisticated analytical tools in financial management is being compounded by the increasing diversity of complex financial instruments and the realization that multiple uncertain factors have a concerted effect on the risk-performance characteristics of securities. The essence of financial management is the study of allocation and deployment of economic resources, both spatially and across time, in an uncertain environment. To capture the influence and dynamic interaction of uncertain risk factors effectively, requires sophisticated analytical tools.

Mathematical models such as the *Stochastic Dominance criteria* include elegant applications of optimization. Over the past decades, financial optimization had a direct and important influence on financial practices. Using mathematical programming techniques, we can reduce financial risks that affect the performance of portfolios, by diversifying away the nonsystematic risk of these portfolios. The diversification principle simply states that the investments should be distributed across various assets so that the exposure to the risk of any particular asset is limited.

The main purpose of this dissertation is to construct optimal portfolios using the concepts of *first-order* stochastic dominance, *second-order* stochastic dominance and *third-order* stochastic dominance as well as the $CVaR$ approach, including three different investment tactics. In particular, we propose to determine the optimal portfolios based on the FSD, SSD and FSD criteria to find the optimal portfolio weights. We implement all the alternative models in the General Algebraic Modeling System (GAMS). In constructing our *FSD-based* portfolio we adopt 0-1 Mixed Integer Linear Programming developed in Kuosmanen (2004), as well as the construction of our *SSD-based* and *TSD-based* portfolios are formulated in terms of standard Linear Programming developed once again in Kuosmanen (2001,2004). Furthermore, in order to compare the performance of the optimal competing portfolios, we evaluate all these alternative portfolios with respect to the market benchmark portfolio using several performance measures such as the Sharpe Ratio, the Sortino Ratio, the opportunity cost and portfolio turnover.

1.2 Literature Review

Stochastic Dominance is a natural setting to use for decision under uncertainty when partial information regarding the decision maker's risk preferences is available. Stochastic Dominance offers criteria to rank two mutually exclusive investments when compared pairwise (Hadar and Russell, 1969, Hanoch and Levy 1969, Levy and Hanoch 1970). Commonly, first order and second order stochastic dominance criteria are being used. Hadar and Russell (1969) and Bawa (1975) show that FSD and SSD amount to choosing the portfolio that maximizes investor's utility assuming that investors preferences are characterized by non-satiation or non-satiation and risk aversion, respectively. Nowadays, the Stochastic Dominance criteria have been established as beneficial analytical tools for studying theoretical probability distributions of random variables in addition to empirical cumulative frequency distributions in practically all areas of Economics.
The theoretical attractiveness of Stochastic Dominance lies in its nonparametric nature. Stochastic Dominance criteria do not require any assumption on the distribution of returns of the two portfolios under consideration and they are consistent with a general class of preferences. Therefore, Stochastic Dominance criteria are a natural candidate to rank two portfolios because they do not impose strict assumptions on preferences and distribution of returns as the commonly used mean-variance portfolio construction approach. Mean-variance approach maximizes expected utility theory only in the case where investor preferences and return distributions obey highly restrictive conditions (i.e. Quadratic utility function and/or normally distributed returns)

The concept of Stochastic Dominance is theoretically appealing: if a return distribution A first-order, second-order, third-order stochastically dominates another distribution B, then all investors with some specific preferences will prefer A to B. Importantly, the Stochastic Dominance criteria also do not focus on a limited number of moments but account for the complete return distribution considering both gains and losses. Thus, the Stochastic Dominance criteria are consistent with the traditional Von Neumann-Morgenstern Expected Utility theory, as well as a wide class of alternative non-expected utility theories.

The optimal investment portfolios is an interesting application field for Stochastic Dominance due to the fact that, first, financial theory does not give us with strong predictions about investor preferences and asset return distributions, and second, nonparametric examination can be improved from large data sets that are now available.

The drawback of the FSD, SSD and TSD criteria though is that they can only compare pairwise any two given portfolios. Hence, they cannot be used to test whether a portfolio stochastically dominates every single portfolio because there is an infinite number of alternative portfolios. Recently, there has been significant progress on computational and statistical issues that have advanced the position of the Stochastic Dominance method, introducing the notion of Stochastic Dominance Efficiency. This notion is a direct extension of Stochastic Dominance to the case where full diversification is allowed. Stochastic Dominance efficiency (SDE) introduced by Post (2003), Kuosmanen (2004), Kopa and Post (2011) as well as Post and Kopa (2013) avoid this constraint. These authors test for Stochastic Dominance of a specified portfolio (the market portfolio) with respect to all other portfolios that can be constructed in a given asset span. Additionally, the test procedures of Kuosmanen (2004) as well as Kopa and Post (2011) identify an efficient portfolio that dominates the evaluated portfolio if the latter is not efficient itself. Moreover they developed linear programming tests for Stochastic Dominance efficiency that do account for diversification possibilities. Although these tests provide an important step in the evolution of the Stochastic Dominance methodology, they rely intrinsically on using ranked observations under i.i.d assumption on the asset returns. Contrary to the initial observations, ranked observations are no more i.i.d. The main limitation of all these works is that they only analyze in-sample performance. For practical portfolio allocation problems, it is important to establish the out-of-sample properties of the Stochastic Dominance efficient portfolios.

Scaillet and Topaloglou (2010) developed consistent tests for Stochastic Dominance Efficiency at any order for time-dependent data. They rely on Kolmogorov-Smirnov type tests inspired by the consistent procedures developed by Barrett and Donald (2003), testing for Stochastic Dominance. In particular, they developed general Stochastic Dominance efficiency tests that compare a given portfolio with an optimal diversified portfolio formed from a given finite set of assets. They build on the general distribution definition of Stochastic Dominance in contrast to the traditional expected utility framework. The Stochastic Dominance Efficiency definition of Scaillet and Topaloglou (2010), consider a portfolio to be Stochastic Dominant efficient when it stochastically dominates all other portfolios for any given Stochastic Dominance Efficient criterion under consideration. If a portfolio dominates all other portfolios then it is not dominated by any other portfolio, thus it is Stochastic Dominance efficient. The Scaillet and Topaloglou (2010) SDE methodology is more general than the previous SDE methodologies in the sense, that it does not assume that asset returns are independent and identically distributed.

1.3 Dissertation Overview

Chapter 2 introduced the theory of how risk-averse investors make choices in a world with uncertainty, as well as the basic concepts of risk measurement and risk management. To provide a framework for analysis where objects of choice are readily measurable, this chapter develops mean-variance objects of choice. This chapter begins with simple measures of risk and return for a single asset and then complicates the discussion by moving to risk and return for a portfolio of many risky assets. Decision rules are then developed to show how individuals choose optimal portfolios that maximize their expected utility of wealth, first in a world without riskless borrowing and lending, and then with such opportunities. Moreover, we lay the foundations for the development of financial optimization models. We include a classification of the risk factors that face an investor in today's financial markets, and then define appropriate risk metrics. The chapter concludes with a presentation of the major risk measures used in portfolio optimization such as variance, mean-absolute deviation, Value at Risk, Conditional Value at Risk, etc as well as the associated mathematical formulations of the optimization models.

Chapter 3 starts with the Utility theory, where we focus on the theory of how people make choices when faced with uncertainty, leading up to the development of the first, second and third stochastic dominance rules (FSD, SSD and TSD, respectively). In particular, we begin with a discussion of the axioms of investor preferences, then used them in order to develop cardinal utility function and finally employ the utility functions to measure risk premium and derive measures of risk aversion. Moreover as the risk premium varies from one investor to another, we conclude that, in general, no one single objective index has the capacity to rank investments by their risk. Thus the whole distribution of returns rather than one measure of profitability and one measure of risk has to be considered. Hence, in this chapter we prove and discuss the stochastic dominance rules stated in terms of cumulative distributions for the partial ordering of uncertain projects.

Chapter 4 introduces notation, defines the key concepts and presents the appropriate mathematical formulations in order to construct optimal portfolios based on the $CVaR$ optimization model, including three different investment tactics, as well as the FSD, SSD and TSD efficiency algorithms developed by Kuosmanen (2001,2004). Furthermore, in this chapter we examine the statistical characteristics of the historical data set, we describe briefly the computational tests and finally we discuss the empirical results.

In Chapter 5 we examine and compare the statistical characteristics of the optimal portfolios, as well as we employ four commonly used parametric performance measures in order to evaluate the performance of all the alternative competing strategies with respect to the market benchmark portfolio: the Sharpe ratio, the Sortino ratio, opportunity cost and portfolio turnover. Chapter 6 concludes the dissertation.

Chapter 2

Modern Portfolio Theory

2.1 Modern Portfolio Theory

The author of the modern portfolio theory is Markowitz (1952) who introduced the analysis of the portfolios of investments in his article "Portfolio Selection". The new approach presented in this article included portfolio formation by considering the expected rate of return and risk of individual stocks and, crucially, their interrelationship as measured by correlation. Prior to this investors would examine investments individually, build up portfolios of attractive stocks, and not consider how they related to each other. Markowitz showed how it might be possible to better of these simplistic portfolios by taking into account the correlation between the returns on these stocks. Moreover, Markowitz portfolio indicates that as we add assets to an investment portfolio the total risk of that portfolio-as measured by the variance (or standard deviation) of the total return-declines continuously, but the expected return of the portfolio is a weighted average of the expected returns of the individual assets. In other words, by investing in portfolios rather than in individual assets, investors could lower the total risk of investing without sacrificing return. Markowitz was the first to clearly and rigorously show how the variance of a portfolio can be reduced through the impact of diversification as he proposed that investors should focus on selecting portfolios based on their overall risk-reward characteristics instead of merely compiling portfolios from securities that each individually has attractive risk-reward characteristics.

The diversification plays a very important role in the modern portfolio theory. The Markowitz Efficient Frontier is the set of all portfolios of which expected returns reach the maximum, given a specific level of risk.

The Markowitz model is based on several presumptions concerning the behavior of investors and financial markets:

- A probability distribution of possible returns over some holding period can be estimated by investors.
- Investors have single-period utility functions in which they maximize their utility within the framework of diminishing marginal utility of wealth.
- Variability about the possible values of return is used by investors to measure risk.
- Investors care only about the means and variance of the returns of their portfolios over a particular period.
- Expected return and risk as used by investors are measured by the first two moments of the probability distribution of returns-expected value and variance.
- Return is desirable; risk is to be avoided. Markowitz model assumes that investors are risk averse. This means that given two assets that offer the same expected return, investors will prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns. Conversely, an investor who wants higher returns must accept more risk. The exact trade-off will differ by investor based on individual risk aversion characteristics. The implication is that a rational investor will not invest in a portfolio if a second portfolio exists with a more favorable risk-return profile.
- Financial markets are frictionless.

Markowitz approach is viewed as a single period approach: at the beginning of the period the investor must take a decision in what particular securities to invest and hold these securities until the end of the period. Because a portfolio is a collection of securities, this decision is equivalent to selecting an optimal portfolio from a set of possible portfolios. Essentiality of the Markowitz portfolio theory is the problem of optimal portfolio selection.

The method that should be used in selecting the most desirable portfolio involves the use of indifference curves. Indifference curves represent an investor's preferences for risk and return. These curves should be drawn, putting the investment return on the vertical axis and the risk on the horizontal axis. Following Markowitz approach, the measure for investment return is expected rate of return and a measure of risk is standard deviation. The map of indifference curves for the individual risk-averse investor is depicted in *Figure 2.1*. Each indifference curve (I_1, I_2, I_3) represents the most desirable investment or investment portfolio for an individual investor. That means, that any of investments (or portfolios) plotted on the indifference curves ($A, B, C\ or\ D$) are equally desirable to the investor.

Features of indifference curves:

- All portfolios that lie on a given indifference curve are equally desirable to the investor. An important implication of this feature is the fact that indifference curves cannot intersect.
- An investor has an infinitive number of indifference curves. Every investor can represent several indifference curves (for different investment tools). Every investor has a map of the indifference curves representing his or her preferences for expected returns and risk (standard deviations) for each potential portfolio.

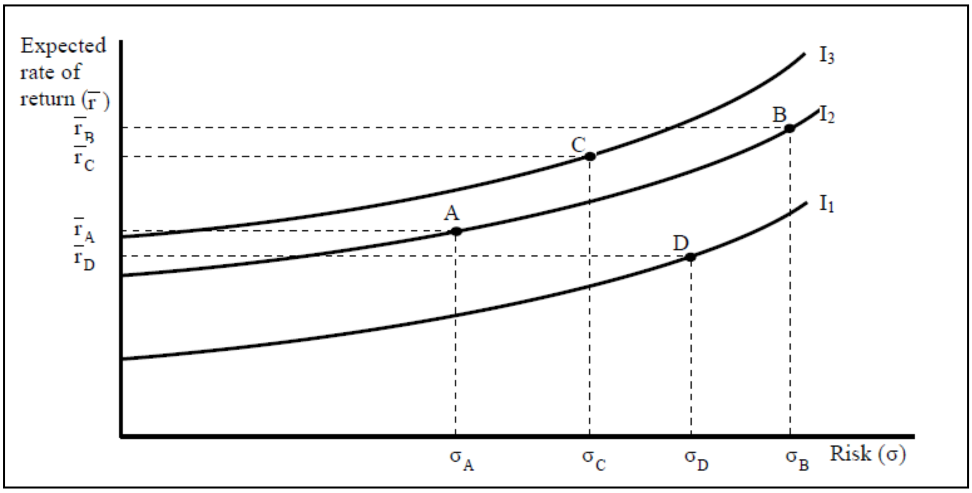

Figure 2.1: Map of Indifference Curves for a Risk-Averse Investor.

Two crucial fundamental assumptions, examining indifference curves, can be applied to Markowitz portfolio theory:

i. The investors are assumed to prefer higher levels of return to lower levels of return, because the higher levels of return allow the investor to spend more on consumption at the end of the investment period. Thus, given two portfolios with the same standard deviation, the investor will choose the portfolio with the higher expected return. This is called an *assumption of nonsatiation.*

ii. Investors are risk averse. It means that the investor when given the choice will choose the investment or investment portfolio with the smaller risk. This is called *assumption of risk aversion.*

2.2 *Measurements of Portfolio Risk and Return*

Modern portfolio theory is a theory on how risk-averse investors can construct portfolios in order to optimize or maximize expected return based on a given level of risk, defined as variance. Its key insight is that an asset's risk and return should not be assessed by itself, but by how it contributes to a portfolio's overall risk and return. According to the theory, it is possible to construct an "efficient frontier" of optimal portfolios offering the maximum possible expected return for a given level of risk.

From this point we assume that investors measure the expected utility of choices among risky assets by looking at the mean and variance provided by combinations of those assets. Unless investors have a special type of utility function (quadratic utility function), it is necessary to assume that returns have a normal distribution, which can be completely described by mean and variance. The normal distribution is perfectly symmetric and 50% of the probability lies above the mean. Because of its symmetry the variance and semivariance are equivalent measures of risk for the normal distribution. Moreover, if we know the mean and standard deviation of a normal distribution, we know the likelihood of every point in the distribution.

2.2.1 *Calculating the Mean and the Variance of a Two-Asset Portfolio*

Consider a portfolio of two risky assets A and B that are both normally distributed. The portfolio mean return is seen to be simply the weighted average of the expected returns on individual securities, where w_A, w_B are the weights of the wealth invested in those assets, respectively.

$$E\left(R_p\right) = w_A E(R_A) + w_B E(R_B) = w_A E(R_A) + (1 - w_A) E(R_B) \qquad (2.1)$$

The variance of the portfolio return is expressed as the sum of the variances σ_A^2, σ_B^2 of the individual securities multiplied by the square of their weights plus a third term, which includes the covariance, $Cov(A, B)$:

$$\sigma_{R_p}^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B Cov(A, B) \qquad (2.2)$$

The covariance is a measure of the way in which the two securities move in relation to each other. If the covariance is positive, this implies that the assets move in the same direction. If it is negative, they move in opposite directions.

2.2.2 *The Correlation Coefficient*

The correlation coefficient is a measure that determines the degree to which two random variables' movements are associated. The range of values for the correlation coefficient is -1.0 to 1.0. The correlation r_{AB}, between two random variables is defined as the covariance divided by the product of the standard deviations:

$$r_{AB} = \frac{Cov(A,B)}{\sigma_A \sigma_B} \qquad (2.3)$$

A correlation of -1.0 indicates a perfect negative correlation, while a correlation of 1.0 indicates a perfect positive correlation. Obviously, if the returns of the two assets are independent, which means that the covariance between them is zero, then the correlation between them will be zero. While the correlation coefficient measures the degree to which two variables are related, it only measures the linear relationship between these variables. Nonlinear relationships between two variables cannot be captures or expressed by the correlation coefficient.

If we rearrange the definition of the correlation coefficient, we will get another definition of covariance whereby it is seen to be equal to the correlation coefficient times the product of the standard deviations:

$$Cov(A,B) = r_{AB}\sigma_A\sigma_B \qquad (2.4)$$

Substituting the previous formulation into the definition of the variance of a portfolio of two assets, we obtain the following expression:

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B r_{AB}\sigma_A\sigma_B \qquad (2.5)$$

2.2.3 *Relationship between correlation coefficients and portfolio variance*

Consider that the two risky assets A and B are perfectly correlated, that is $r_{AB} = 1$, then the portfolio variance becomes:

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B Cov(A,B) \Rightarrow$$
$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B \sigma_A\sigma_B \Rightarrow \sigma_{R_p}^2 = (w_A\sigma_A + w_B\sigma_B)^2$$

therefore the standard deviation will be

$$\sigma_{R_p} = w_A\sigma_A + w_B\sigma_B \quad (2.6)$$

Figure 2.2 depicts the portfolio mean and standard deviation on a single graph. Point A and point B represent the risk and return for a portfolio consisting of 100% of our investment in security A and in security B respectively. The dashed line represents the risk and return provided for all combinations of A and B when the returns of these two assets are perfectly correlated.
Suppose now that the returns on A and B are perfectly inversely correlated; in other words, $r_{AB} = -1$, then the portfolio variance becomes:

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_BCov(A,B) \Rightarrow$$
$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 - 2w_Aw_B\sigma_A\sigma_B \Rightarrow \sigma_{R_p}^2 = (w_A\sigma_A - w_B\sigma_B)^2$$

therefore the standard deviation will be

$$\sigma_{R_p} = \pm(w_A\sigma_A - w_B\sigma_B) \qquad (2.7)$$

In this case the graph of the relationship between mean and standard deviation is the dotted line ABC, which is two line segments, one with a positive slope and the other with a negative slope.
Finally, assume that the returns on the two assets are independent, which implies that $r_{AB} = 0$, then the portfolio variance becomes:

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_BCov(A,B) \Rightarrow$$
$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2$$

as well as the standard deviation can be expressed as:

$$\sigma_{R_p} = \sqrt{(w_A^2\sigma_A^2 + w_B^2\sigma_B^2)} \qquad (2.8)$$

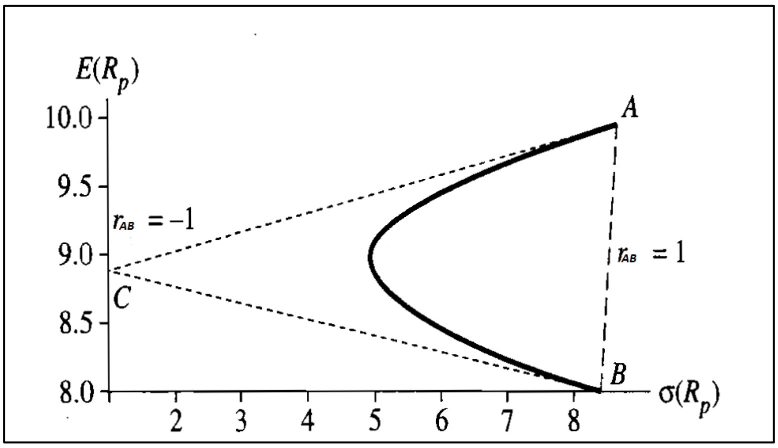

Figure 2.2: Trade-off between mean and standard deviation.

In general, the variance of the portfolio can be expressed as

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_A w_B Cov(A, B)$$

if we take into consideration that the sum of weights must add to 1 and replace $w_B = 1 - w_A$, can be written as follows:

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A(1 - w_A)Cov(A, B) \Rightarrow$$

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + \sigma_B^2 - 2w_A\sigma_B^2 + w_A^2\sigma_B^2 + 2w_A Cov(A, B) - 2w_A^2 Cov(A, B) \quad (2.9)$$

2.3 Computing the minimum variance portfolio

The previous formula of the variance of the portfolio can be used in order to find the combination of random variables A and B, that provides the portfolio with minimum variance. This portfolio is the one where changes in variance or standard deviation with respect to changes in the proportion of wealth invested in security A are zero. We can minimize portfolio variance by setting the first derivative equal to zero:

$$\frac{d\sigma_{R_p}^2}{dw_A} = 0 \Rightarrow 2w_A\sigma_A^2 - 2\sigma_B^2 + 2w_A\sigma_B^2 + 2Cov(A, B) - 4w_A Cov(A, B) = 0$$

Solving for the optimal proportion to invest in security A in order to obtain the minimum variance portfolio we get the following formula

$$w_A^* = \frac{\sigma_A^2 - Cov(A, B)}{\sigma_A^2 + \sigma_B^2 - 2Cov(A, B)} \quad (2.10)$$

Line AB in *Figure 2.2* represents the risk-return trade-offs available to the investors if the two assets are perfectly correlated, and line segments AC and AB show the trade-offs if the assets are perfectly inversely correlated. The general slope of the mean-variance opportunity set is the solid line. In general the minimum variance opportunity set is the region of risk and return combinations offered by portfolios of risky assets that yields the minimum variance for a given rate of return. This minimum variance opportunity set will be convex due to the fact that the opportunity set is bounded by the triangle ACB.

2.4 The concept of Markowitz efficient frontier

Every possible asset combination can be plotted in risk-return space, and the collection of all such possible portfolios defines a region in this space. The line along the upper edge of this region is known as the efficient frontier. Combinations along this line represent portfolios for which there is lowest risk for a given level of return. Conversely, for a given amount of risk, the portfolio lying on the efficient frontier represents the combination offering the best possible return. Mathematically the efficient frontier is the intersection of the set of portfolios with minimum variance and the set of portfolios with maximum return.

Figure 2.3 represents investors entire investment opportunity set, which is the set of all attainable combinations of risk and return offered by portfolios formed by asset A and asset B in different proportions. The curve passing through *A* and B shows the risk-return combinations of all the portfolios that can be formed by combining those two assets. Investors desire portfolios that lie to the northwest in *Figure 2.3*. These are portfolios with high expected returns and low volatility.

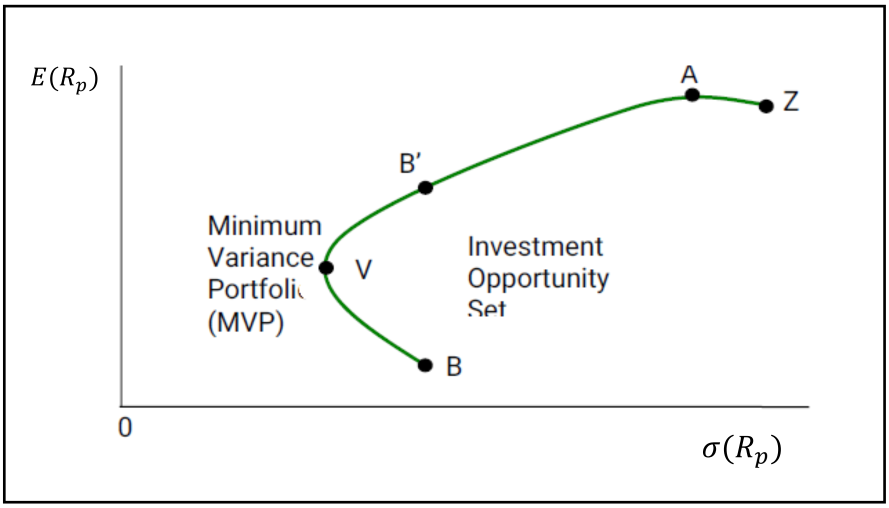

Figure 2.3: Investment Opportunity set for assets A and B.

The area within curve *BVAZ* is the feasible opportunity set representing all possible portfolio combinations. Portfolios that lie below the minimum-variance portfolio point (*V*) on the figure can therefore be rejected out of hand as inefficient. The portfolios that lie on the frontier *VB* in *Figure 2.3* do not meet the criteria of maximizing expected return for a given level of risk or minimizing risk for a given level of return. This is easily seen by comparing the portfolio represented by points *B* and *B’*. Since investors always prefer more expected return than less for a given level of risk, *B’* is always better than *B*. Using similar reasoning, investors would always prefer V to B because it has both a higher return and a lower level of risk. In fact, the portfolio at point *V* is identified as the minimum-variance portfolio; since no other portfolio exists that has a lower standard deviation. The curve *VA* represents all possible efficient portfolios and is the efficient frontier, which represents the set of portfolios that offers the highest possible expected rate of return for each level of portfolio standard deviation. As we can observe, the efficient frontier will be convex, due to the fact that the risk-return characteristics of a portfolio change in a non-linear way as its component weightings are changed. The efficient frontier is a parabola when expected return is plotted against variance (standard deviation).

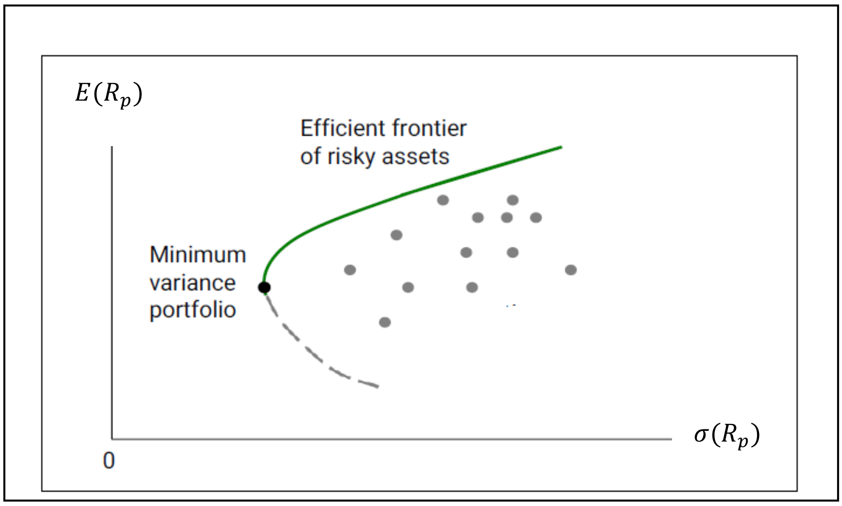

Figure 2.4: The efficient frontier of risky assets.

As we can see in *Figure 2.4* any portfolio on the downward sloping potion of the frontier curve is dominated by the portfolio that lies directly above it on the upward sloping portion of the frontier curve since that portfolio has higher expected return and equal standard deviation. The best choice among the portfolios on the upward sloping portion of the frontier curve is not as obvious, because in this region higher expected return is accompanied by higher risk. The best choice will depend on the investor's willingness to exchange risk against expected return.

2.5 The Efficient Frontier with Two Risky Assets and No Risk-Free Asset

The hypothesis of no risk-free asset is the same as saying that we do not consider any borrowing or lending opportunities. An investor will select his optimal portfolio of risky assets in an economy where there is no opportunity for exchange. Based on the utility theory we know that the indifference curves for the risk-averse investors are convex in the mean-variance plane. In this particular analysis, we assume that investors have similar beliefs about the opportunity set, that the existence of a risk-free asset is permitted, and that the investors have different indifference curves, which reflect their different behavior towards risk. *Figure 2.5* represents three different indifference curves, as well as the investment opportunity set. As we can observe investor Ⅲ is more risk averse than investor Ⅱ who in sequence is more risk averse than investor Ⅰ. Therefore, they each will prefer to invest a different proportion of their portfolio wealth in the risky assets that create the opportunity set. Moreover all rational investors will never prefer a portfolio below the minimum variance point. They can always attain higher expected utility along the positively sloped portion of the opportunity set represented by the line segment $EDCBA$. This approach brings on the definition of the efficient set. The efficient set is the set of mean-variance choices from the investment opportunity set where for a given level of variance no other investment opportunity offers a higher mean return. The concept of an efficient set eliminates the number of portfolios from which an investor may choose. In *Figure 2.5* the portfolios at points B and F offer the same standard deviation, but B is on the efficient set due to the fact that it provides a higher return for the same level of risk. Therefore no rational investor would ever select point F over point B and for that reason we can avoid point F.

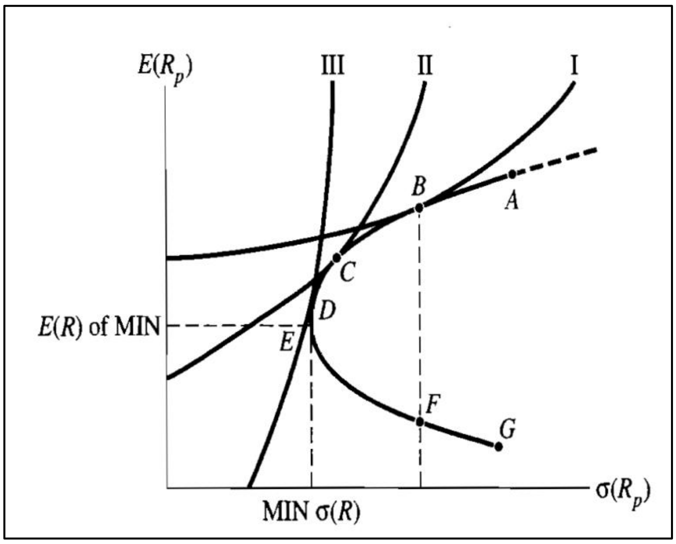

Figure 2.5: Choices by investors with different indifference curves.

In general the region of feasible mean-variance opportunities can be found by solving either of the following two mathematical programming models. The first emphasizes the minimum variance opportunity set and the second determines the efficient set.

Mathematical Model A:

$$min\{\sigma_{R_p}^2 = [w_A^2\sigma_A^2 + (1-w_A)^2\sigma_B^2 + 2w_A(1-w_A)r_{AB}\sigma_A\sigma_B]\} \qquad (2.11.a)$$

subject to

$$E(R_p) = w_A E(R_A) + (1 - w_A)E(R_B) \ = K. \qquad (2.11.b)$$

Mathematical Model B:

$$\max\{E(R_p) = [w_A E(R_A) + (1 - w_A)E(R_B)]\} \qquad (2.12.a)$$

subject to

$$\sigma_{R_p}^2 = w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A(1 - w_A)r_{AB}\sigma_A\sigma_B = K. \qquad (2.12.b)$$

The minimum variance opportunity set derives by finding all the combinations that yield the lowest risk for a given level of return. The efficient set is the region of highest returns for a given level of risk.

We can observe that the first problem is a quadratic programming model since the objective function consists of squared terms in the variable w_A. Markowitz [1959] was the first to describe the investor's portfolio decision problem in this way and to show that it is the same as to maximizing the investor's expected utility.

2.6 The efficient Set with One Risky and One Risk-Free Asset

Consider now the case where one of the two assets, R_f, that is the risk-free asset, has zero variance, then the mean and variance of the portfolio become:

$$E(R_p) = aE(R) + (1 - a)E(R_f) \qquad (2.13)$$

$$\sigma_{R_p}^2 = a^2\sigma_X^2 \qquad (2.14)$$

The variance and the covariance of the risk-free asset with the risky asset are zero; therefore the variance of the portfolio is simply the variance of the risky asset. *Figure 2.6* depicts the opportunity set with one risk-free and one risky asset. We can observe that this is a straight line.

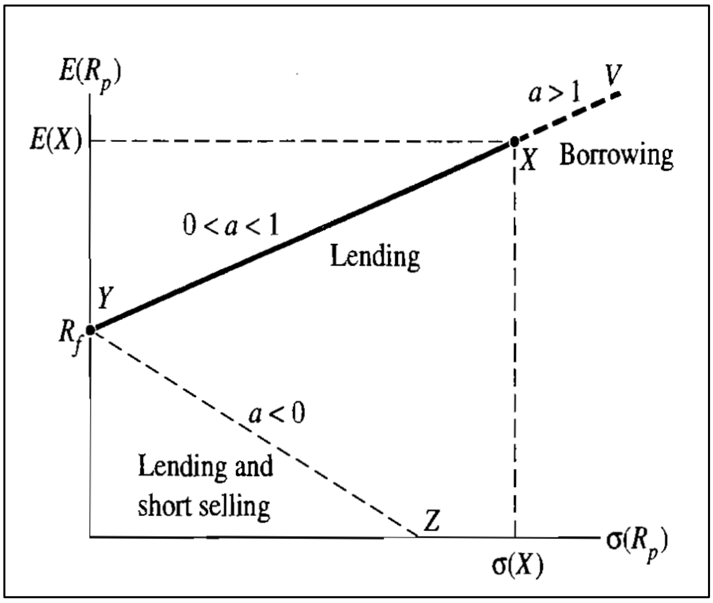

Figure 2.6: Opportunity set with one risky and one risk-free asset.

Provided for the supposition that the borrowing rate is equivalent to the lending rate, YXV is a straight line. In order to purchase portfolios along the line segment XV, it is essential to borrow in order to invest more than 100% of the portfolio in the risky asset, hence along the line segment XV the percentage invested in asset X is greater than 1; that is to say, $a > 1$. Moreover, when we decide to invest more than 100% of our portfolio in the risk-free asset, we should sell short the risky asset. Therefore the line segment YZ represents portfolio mean and variance in this case. As for the efficient set which is the positively sloped line segment XYV, is composed of long positions in the risky asset combined with borrowing or lending.

2.7 Portfolio Mean, Variance, and Covariance with N risky assets

An investor can reduce portfolio risk by holding combinations of instruments that are not perfectly positively correlated. In other words, investor can reduce their exposure to individual asset risk by holding a well-diversified portfolio of assets. Diversification may allow for the same portfolio expected return with reduced risk. These ideas have been started with Markowitz and then reinforced by other economists and mathematicians such as Andrew Brennan who have expressed ideas in the limitation of variance through portfolio theory.

Consider now the case where we wish to discuss about the mean and the variance of a portfolio consisted of N assets instead of just two. Hence, the expected portfolio return is simply the weighted average of the expected return on individual securities. This can be written as

$$E(R_p) = \sum_{i=1}^{N} w_i E(R_i) \quad (2.15)$$

where R_i is the return on asset i and w_i is the proportion of wealth invested in each asset i.

The portfolio variance is a weighted sum of variance and covariance terms. This can be expressed as

$$\sigma_{R_p}^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \quad (2.16)$$

where w_i and w_j are the percentages invested in each asset and σ_{ij} is the covariance of asset i with asset j. The previous formula can also be written as

$$\sigma_{R_p}^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j r_{ij} \quad (2.17)$$

where σ_i, σ_j are the standard deviations of assets i and j ,respectively , as well as r_{ij} is the correlation coefficient between these securities.

Equations (2.15) and (2.16) can also be written in terms of matrices form, which for N assets looks like:

$$E(R_p) = \boldsymbol{R}'\boldsymbol{W} \qquad (2.18)$$

$$\sigma_{R_p}^2 = \boldsymbol{W}' \boldsymbol{\Sigma} \boldsymbol{W} \qquad (2.19)$$

The expected portfolio return is the $(1 \times N)$ row vector of expected returns, $\boldsymbol{R}'$, postmultiplied by the $(N \times 1)$ column vector of weights held in each asset, $\boldsymbol{W}$. The variance is the $(N \times N)$ variance-covariance matrix, $\boldsymbol{\Sigma}$,premultiplied and postmultiplied by the vector of weights , $\boldsymbol{W}$.

Finally suppose that we want to express the covariance between two portfolios, A and B, using matrix notation. If $\boldsymbol{\Sigma}$ is the $(N \times N)$ variance-covariance matrix, then the covariance between these two portfolios is defined as

$$Cov(R_A, R_B) = \boldsymbol{W}_1' \sum \boldsymbol{W}_2 \qquad (2.20)$$

Where $\boldsymbol{W}_1'$ is the $(1 \times N)$ row vector of weights held in portfolio A and $\boldsymbol{W}_2$ is the $(N \times 1)$ column vector of weights used to construct portfolio B.

2.8 The Opportunity Set with N Risky Assets

With regard to the construction of portfolios with many assets, we can find the opportunity set and the efficient frontier if we know the expected returns and the variances of individual assets as well as the covariances between each combination of assets. The investment opportunity set has the same shape with many risky assets as it did with only two assets. The main difference is that with many assets to be under consideration some of them will fall in the interior of the opportunity set (*Figure 2.7*). The opportunity set will be consisted of various portfolios and of some individual assets that are mean-variance efficient by themselves. Under these circumstances a risk-averse investor would maximize his expected utility by finding the point of tangency between the efficient set and the highest indifference curve.

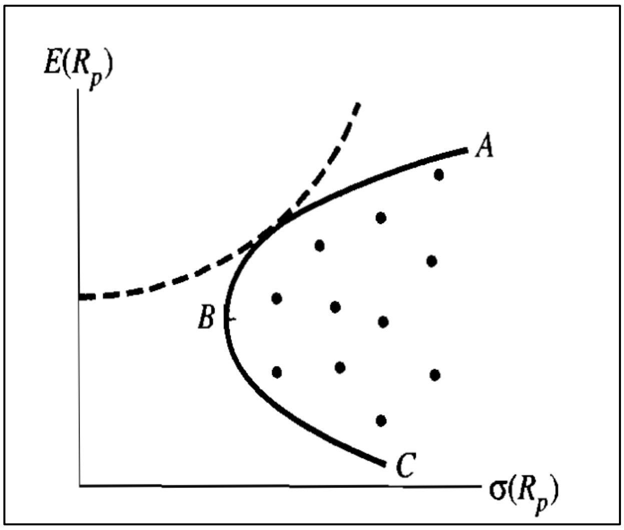

Figure 2.7: The investment opportunity set with many risky assets.

2.9 The Efficient Frontier with N Risky Assets and One Risk-Free Asset

We have already mentioned that the introduction of a risk-free asset may be thought of as creating an exchange of market economy where there are many individuals. Each of them may borrow or lend unlimited amounts at the risk-free rate. If in addition to the previous presumption of equality between the borrowing and lending rate, we add the assumption that all investors have similar beliefs about the expected distributions of returns offered by all assets, then all investors will perceive the same linear efficient set called the Capital Market Line. *Figure 2.8* is a graph of the Capital Market Line. The intercept is the risk-free rate, R_f, and its slope is $[E(R_m) - R_f]/\sigma(R_m)$. Hence, the equation defining the Capital Market Line is the following.

$$E(R_p) = R_f + \frac{[E(R_m) - R_f]}{\sigma(R_m)}\sigma(R_p) \qquad (2.21)$$

The previous formula provides a simple linear relationship between the risk and return for the efficient portfolios of assets.

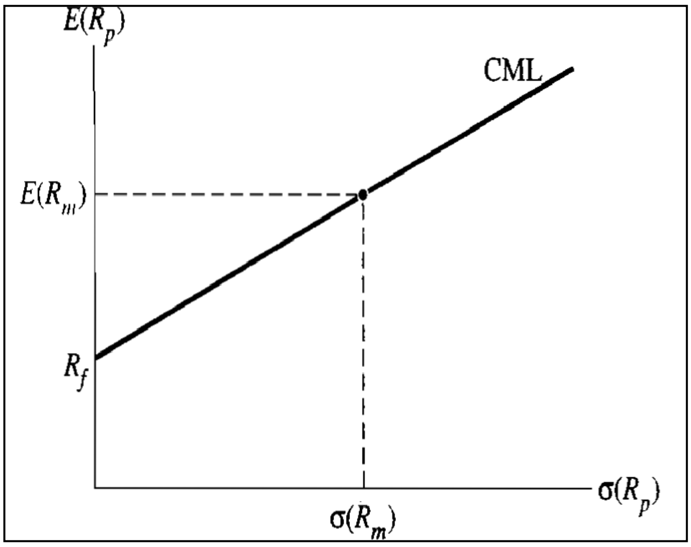

Figure 2.8: The Capital Market Line.

Finally in *Figure 2.9* feasible and efficient sets of portfolios are presented. Considering the assumptions of nonsatiation and risk aversion discussed earlier in this section, only those portfolios lying between points A and B on the boundary of feasibility set investor will find the optimal ones. All the other portfolios in the feasible set are inefficient portfolios. Furthermore, when a risk-free investment is introduced into the universe of assets, the efficient frontier becomes the tangential line shown in *Figure 2.9* and the portfolio at the point at which it is tangential (point *M*) is called the Market Portfolio.

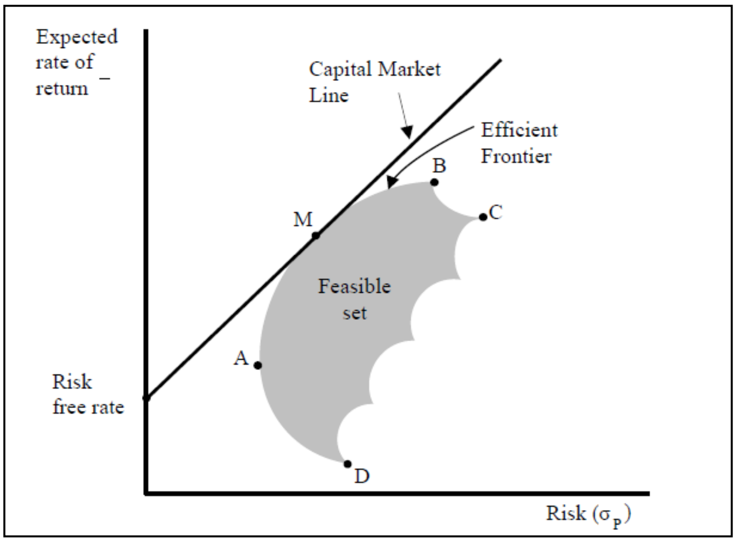

Figure 2.9: Feasible set and Efficient set of Portfolios.

Basic Ideas in Risk Management

2.10 Basic Ideas in Risk Management

Risk is the degree of uncertainty in attaining a certain level of portfolio return. It reflects the chance that the actual return of the portfolio may be very different than the expected return. Risk in international portfolios arises due to uncertainties in the return of the assets (market risk) and the exchange rates (currency risk). Investors can neither ignore nor insure themselves completely against these risks. They must be aware of their exposures to these risk factors and take them into consideration in their decision process so as to properly manage their total level of risk.

The liberalization of markets and their consequent interdependencies, increasing complexity of innovative financial instruments, intensifying competition, regulatory environments and the realization of severity of potential losses dictate the development of integrated portfolio risk management models that take into account all the needs and control simultaneously the exposure of international portfolios to different risk factors. Derivative securities (forwards, stock and currency options) provide appropriate means to hedge the multiple risks.

Risk means the danger of loss. If we have an exposure to a risk, it means that there are some circumstances under which we can lose money, some hypothetical future loss. Moreover, financial risk is the possibility that an unforeseen and unpredictable future event will result in a financial loss. Financial risk may be characterized by the magnitude of the loss, its estimated likelihood today, and the causes of the event which are known as risk factors. A classification of financial risk factors, the potential causes of loss, is given in the following section. Risk is always in the future, as current or past losses do not present a risk as there is no uncertainty about them. Future events do not pose a risk unless they are unpredictable, for otherwise we could plan for them with perfect foresight. However the unpredictability of the future does not restrict our ability to foresee plausible future events and plan for them. Risk management is the discipline that provides tools to measure risks, and techniques to help us shape and make rational decisions about them. Risk management is not restricted to financial institutions, and furthermore the distinction between financial and other types of risks is becoming increasingly blurred. To manage risks, we must first understand what is risk, what are the different types of risk, and how to measure them.

2.10.1 A Classification of Financial Risks

As we have already mentioned financial risk is the uncertainty surrounding the value of our assets due to unforeseen and unforecastable future events. The causes of uncertainty are many, and financial risk is multidimensional. Consider that there are n available financial assets from which we can construct a portfolio by investing a proportion of our wealth in each asset. The portfolio return is uncertain and therefore risky. It is subjected to the unforeseen and unpredictable changes that may occur to the assets returns. The sources of financial risks are the forces that influence the returns of the assets in our portfolio. We display here a classification of these risk factors highlighting the fact that financial risk is multidimensional.

- <u>Market Risks</u>: These are the risks arising from changes in financial market prices and rates. Moreover is the risk that the market price of the assets will change with time. Depending on the type of asset we distinguish between stock market price risk and fixed income market price risk.
    - i. *Stock Market Price Risk:* The risk that the price of a stock will change with time due to adverse movements of the stock market as reflected in market index changes.
    - ii. *Fixed Income Market Price Risk:* The risk that the price of a fixed-income security will change with time due to adverse movements of the fixed income market as reflected in market index changes. The prevailing risk in fixed income markets is the risk caused by

movements in the overall level of interest rates on straight, default-free securities.

iii. *Interest Rate Risk:* The risk that the price of a security will change with time due to movements of the general level of interest rates.

iv. *Shape Risk:* The risk that the price of a security will change with time due to changes in the shape of the term structure of interest rates.

v. *Volatility Risk:* The risk that the price of an asset will change with time due to changes in volatility. This type of risk is predominant in options. If the underlying asset is worth more than the strike price of a call option on the expiration date then the option is exercised returning to its holder the difference between the price of the asset and the strike price. But if the price of the asset is less than the strike price then the option expires worthless. The higher the price volatility of the underlying asset the more valuable is the option. Therefore, volatility changes have an impact on the prices of options or securities with embedded options even in an environment that remains unchanged in all other respects.

- Credit Risk: The risk of an unkept payment promise due to default of an obligor (counter-party, issuer or borrower) or due to adverse price movements of an asset caused by an upgrading or downgrading of the credit quality of an obligor that brings into question their ability to make future payments. Credit risk is the risk of loss from the failure of the counterparty to fulfill its contractual obligations, perhaps because they have defaulted. The magnitude of the loss can be gauged by the free market cost of replacing the lost cashflow or cashflows. Credit risk occurs in a lot of settings:
    - Loans, where we lend to a corporation money on a bilateral basis, expecting them to make payments of interest and to repay principal.
    - Contractual agreement such as IRSs or purchased options where our counterparty either certainly has to make payments in the future (as in the swap) or may have to (if we exercise an option they have written).
    - Receivables, where goods are delivered or services performed before they have been paid for.
- Currency Risk: The risk that the price of a security will change with time due to changes in the exchange rates between different currencies. Investors with international holding are exposed to the risk of fluctuations of the exchange rates of their base currency vis-à-vis the currencies of countries where they hold assets.
- Liquidity Risk: Liquidity is the ability to meet expected and unexpected demands for cash. This type of risk therefore, is the risk that we will not be able to do that- that we will face the requirement to pay cash and be unable

to do so. Liquidity risk can occur when we have more assets than liabilities, but when we are unable to liquidate those assets in time. This is an important risk class for many financial institutions precisely because they often have illiquid assets and more liquid liabilities.

- Sector Risk: The risk of price movements affecting a group of securities that share common characteristics. Sectors of the economy are affected by different macroeconomic conditions and other factors.
- Residual Risk: This is the risk of price movements due to firm-specific effects and in principle, unrelated to the systematic influences given in our list of risks. Mergers and acquisitions, or corporate strategy are sources of residual risk. When an industrial segment is undergoing some transformation, then mergers and acquisitions may result into sector risk.
- Business Risks: These are the risks due to volatility of volumes, margins, or costs when engaging in the firm's business. For firms which are in the business of selling insurance risks can be made more precise though the definition that follows. One could define similarly, the business risks for other enterprises- both financial institutions and other firms. However, insures have a long tradition in the management of risk, and for their business a widely accepted definition of business risk is currently available.
- Actuarial Risk: This is the risk associated with the liability side of the balance sheet of insurance firms, and is due to changes in mortality, casualty or liability exposures. In a way actuarial risk is a non-financial risk. However, several innovative insurance products provide a combination of an insurance policy and an endowment and it may be hard to disentangle the actuarial risk as defined above from the relevant financial risk.
- Operational Risk: This is the risk of direct or indirect losses resulting from inadequate or failed internal processes, people and systems, and from external events. This type of risk captures all non-financial risks, starting from the risks due to a breakdown in risk management operations and moving on to environmental risks. The breakdown of the risk management operations may be due to miscalculation of market prices or hedging strategies, system failures or faulty controls that leave the institution unable to execute its transactions and fulfill its obligations, human error and fraud, or management failures. Operational risks are characterized by low probability events that result into extreme financial losses.
- Systemic Risk: The risk of a wide-spread collapse or dis-functioning of the financial markets through multiple defaults; widespread disappearance of liquidity, domino effects etc.

2.10.2 *Risk Measures*

The application of effective decision support tools for risk management must employ risk measures that properly reflect and quantify the risk exposure of an investor. The risk management models must be able to flexibly incorporate different risk measures when this is required by regulators or management preferences.

The systematic risk is predominantly the market risk. Hence, one way to measure the risk of an asset is to measure its sensitivity to the changes in the market index. This sensitivity is captured by the *beta* of the security. The *beta* of an asset i is defined by

$$\beta_\iota = \frac{\sigma_{\iota M}}{\sigma_M^2} \qquad (2.22)$$

where $\sigma_{\iota M}$ is the covariance of the random variable asset rate of return r_i and the market rate of return r_M, and σ_M^2 is the variance of the market rate of return. The beta of a security measures the sensitivity of the expected return of the security to changes in a broad market index.

When using optimization models for risk management, one way to measure the risk of an equity portfolio is either directly-using the moments of the distribution of the asset's price r_i to estimate moments of the portfolio return-or indirectly-using the *beta* of the assets in the portfolio. If we view the portfolio as just one more asset we obtain *the beta of the portfolio* as follows; The *beta of a portfolio* of holdings x_i in asset i is given by:

$$\beta_p = \sum_{i=1}^{n} \beta_i x_i \qquad (2.23)$$

The *beta of the portfolio* measures the sensitivity of the portfolio return to changes in the market index. In this sense beta is a measure of risk. A *zero-beta* portfolio is one that remains invariant with changes in the index, while a *beta-neutral* portfolio is one with $\beta_p = 1$ so that portfolio returns change just like the returns of the index.

Asset allocation is a problem faced by every investor. When making investment decisions, an investor has to seek a balance between risk and returns. In the single-period Markowitz model, the investor maximizes the expected return of the portfolio and minimizes the risk, measured by the variance of portfolio returns. The variance of a portfolio of holdings x_i in asset i is given by:

$$\sigma(x)^2 = \sum_{i=1}^{n} \sum_{i=1}^{n} \sigma_{ii'} x_i x_{i'} \qquad (2.24)$$

where $\sigma_{ii'}$is the covariance between the returns of assets i and i', and we have $\sigma_{ii} = \sigma_i^2$. Using the portfolio variance as the risk measure has its limitations. The variance is a symmetrical measure that does not take into consideration the direction of movement. The square root of variance gives us the standard deviation. Standard deviation, also known as volatility, has been the most widely used measure of risk. However, this measure relies on the assumption that the portfolio return distribution is symmetric and implies that the sensitivity of the investor is the same on the upside as on the downside. In order to take the asymmetry of the portfolio return distribution into consideration, the use of downside measures has been advocated.
Considering the control of the overall portfolio risk, we need to employ risk measures that account for this asymmetry of portfolio returns. In order to address this issue, alternative risk measures such as *Value at Risk* and *Conditional Value at Risk* have been introduced to replace the variance.

2.11 Scenario Generation

A general approach to define risk is by using scenarios. Each scenario is a realization of the future value of all parameters that affect the performance of the portfolio under consideration. The collection of scenarios captures the range of likely variations in these parameters that could happen between the current time and the end of the planning horizon. These representations of uncertainty are fundamental in risk management. Scenarios can capture the disparate sources of risk and empower measures to be developed that record for all sources of risk.
Scenario generation is a critical step in the modeling procedure. A set of representative scenarios is required that sufficiently illustrates the expected evolution of the underlying financial primitives and is consistent with the market observations and financial theory. Moreover, scenarios ought to be derived from a “correct`` theoretical model for the random variables, catching at the same time the applicable previous history. Therefore, they should precisely approximate the theoretical model from which they are inferred; a substantial number of scenarios might be important, utilizing a fine discretization method. The scenarios should satisfy the no-arbitrage properties. This implies that scenario-based estimates of future asset prices in a portfolio optimization model should not permit arbitrage opportunities.
The origins of scenarios can be excessively varied. They might be acquired from a discrete known distribution, be obtained in the course of a discretization of some continuous probability distribution which is estimated from historical data, from

economic forecasting models, from bootstrapping historical market data, they can be augmented with subjective opinions of experts etc. The main crucial step is to outline the structure of the scenario tree, which is the number of stages and the branching scheme. The stages relate to points in time when it is possible to take additional decisions in view of recently observed data. Such data can be obtained at specific dates or at regular intervals.

Various scenario generation approaches have been accounted for in literature. One method is to create scenarios by bootstrapping past market observations of asset returns. The major advantage of this approach is its absence of complication. It presumes that past market conditions precisely depict the plausible joint outcomes of the arbitrary variables in the future. It also assumes that these past observations of asset returns are samples from independent and identical distributions. Bootstrapping captures co-movements of numerous random variables, but cannot perform well in temporal dependencies.

Another method that can be used as a part of the scenario generation is the statistical analysis of historical market data. Market data are in the form of correlated, multivariate time series. The dimensionality of the random variables can be diminished with techniques of multivariate statistics. Factor analysis is generally applied under the normality assumption. Principal component analysis can be applied for arbitrary empirical distributions. Both of these methods intend to clarify the correlation structure of the multivariate random variables by a small number of uncorrelated factors or components.

A moment-matching approach creates scenarios so that principal moments of the random variables correspond to particular target values. Given the empirically observed statistical characteristics of the random variables, we make use of a scenario generation procedure that does not assume any specific distributional formation. This method produces a set of scenarios so that selected statistical properties of the random variables match specified target values. In particular, we match the following statistics: the first four marginal moments such as mean, variance, skewness and kurtosis, as well as the correlations of the monthly asset returns.

Finally another scenario generation alternative is to examine from continuous distributions or stochastic procedures for the fundamental financial variables. The assumed distributions are normally adjusted utilizing empirical market data. The precision of the subsequent scenario sets relies on the coarseness of the discretization. Better discretizations reflect more precisely the continuous distributions, but lead to a large number of scenarios and very large-scale stochastic programs. In addition to all these, a balance is required between the accuracy of the discretization and the computational tractability of the resulting stochastic program.

2.12 Optimization Models

Optimization models for financial risk management often take the following form: an appropriate risk measure is optimized subject to operating constraints and a parametric constraint that a desirable performance measure (such as expected portfolio return) meets a prespecified target level. For a given confidence level, a *CVaR* constraint is tighter than a *VaR* constraint if the *CVaR* and *VaR* bounds coincide. By optimizing *CVaR* we minimize the conditional expectation of portfolio losses in excess of a prespecified percentile of the return distribution. Our motivation for applying *CVaR* models stems from the observation that returns of international assets and proportional changes of exchange rates are not normally distributed; they exhibit asymmetric distributions with fat tails. Moreover, we incorporate derivative securities in the optimization models. The non-linear payoff characteristic of options leads to asymmetric portfolio returns distributions. *CVaR* is a coherent risk measure and is suitable for asymmetric distributions.

Consider a set of investment opportunities indexed by $i = 1,2, \dots, n$. At the end of a certain holding period the assets generate returns $\tilde{\boldsymbol{r}} = (\tilde{r_1}, \tilde{r_2}, \dots, \tilde{r_n})^{\mathsf{T}}$. The returns are unknown at the beginning of the holding period and are treated as random variables. Denote their mean value by $\bar{\boldsymbol{r}} = \boldsymbol{\mathcal{E}}(\tilde{\boldsymbol{r}}) = (\overline{\mathrm{r}_1}, \overline{\mathrm{r}_2}, \dots, \overline{\mathrm{r}_\mathrm{n}})^{\mathsf{T}}$. At the beginning of the holding period the investor wishes to apportion his budget to these assets by deciding on a specific allocation $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^{\mathsf{T}}$, such that $x_i \geq 0$ (short sales are disallowed) and $\sum_{i=1}^{n} x_i = 1$ (budget constraint). Using the conformable vector **1** = $(1,1,\dots,1)^{\mathsf{T}}$ of ones, we express the basic portfolio constraints in vector notation as $X = \{\boldsymbol{x}: \boldsymbol{x}^{\mathsf{T}}\mathbf{1}=1,\ \mathbf{x} \geq 0\}$. We use boldface characters to denote vectors and $\sim$ to denote random variables. The uncertain return of the portfolio at the end of the holding period is $\boldsymbol{R}(\boldsymbol{x}, \tilde{\boldsymbol{r}}) = \boldsymbol{x}T\tilde{\boldsymbol{r}} = \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} x_{\boldsymbol{i}} \tilde{r}_{\boldsymbol{i}}$. This is a random variable with a distribution function, say F, i.e. $F(\boldsymbol{x}, u) = P\{R(\boldsymbol{x}, \tilde{\boldsymbol{r}}) \leq u\}$. Of course the distribution function F depends on the portfolio composition $\boldsymbol{x}$. The expected return of the portfolio is $\boldsymbol{\mathcal{E}}(R(\boldsymbol{x}, \tilde{\boldsymbol{r}})) = R(\boldsymbol{x}, \bar{\boldsymbol{r}}) = \boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{r}}$. Suppose the uncertain returns of the assets, $\tilde{\boldsymbol{r}}$, are represented by a finite set of discrete scenarios $\Omega = \{s: s = 1,2, \dots, S\}$, whereby the returns under a particular scenario $s \in \Omega$ take the values $\boldsymbol{r_s} = (r_{1s}, r_{2s}, \dots, r_{ns})^{\mathsf{T}}$ with associated probability $p_s > 0, \sum_{s=1}^{S} p_s = 1$. The mean return of the assets is $\bar{\boldsymbol{r}} = \sum_{s=1}^{S} p_s \boldsymbol{r}_s$. The portfolio return under a particular realization of asset returns $\boldsymbol{r_s}$ is denoted $\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{r_s}) = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{r_s} = \sum_{i=1}^{n} x_i r_{is}$. The expected portfolio return is expressed as $\boldsymbol{R}(\boldsymbol{x}, \bar{\boldsymbol{r}}) = \sum_{s=1}^{S} p_s \boldsymbol{R}(\boldsymbol{x}, \boldsymbol{r}_s) = \boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{r}} = \sum_{i=1}^{n} x_i \bar{r}_i$.

Suppose φ is some risk measure. Then for a certain minimal expected portfolio return μ, the φ -efficient portfolio is obtained from the solution of the following problem:

$$Minimize\ x \in X\ \ \varphi(\boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{r}})$$

$$s.t.\ \boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{r}} \geq \mu \quad (2.25)$$

The curve that depicts the dependence of the optimal value of this parametric program on the required minimal expected portfolio return μ is the φ-efficient frontier. This is a generalization of the classical concept of the mean-variance efficient frontier to an arbitrary risk measure φ. The choice of the risk measure generally depends on the preferences of the decision maker or, in some cases, on regulatory specifications. Matters of computational tractability also affect this choice.

2.12.1 Coherent Risk Measures

A coherent risk measure φ is a function that assigns numbers $\varphi(\tilde{X}), \varphi(\tilde{Y})$ to two random variables $\tilde{X}\ and\ \tilde{Y}$ independent or not and for each number n and positive number λ the following relations hold:

Sub -additivity. $\varphi\left(\tilde{X}+\tilde{Y}\right) \leq \varphi\left(\tilde{X}\right)+\varphi(\tilde{Y})$
Homogeneity. $\varphi\left(\lambda\tilde{X}\right) = \lambda\varphi\left(\tilde{X}\right)$
Monotonicity. $\varphi\left(\tilde{X}\right) \leq \varphi\left(\tilde{Y}\right)\ if\ \tilde{X} \leq \tilde{Y}$
Risk-free condition. $\varphi\left(\tilde{X}-nr_f\right) = \varphi\left(\tilde{X}\right)-n$

Sub-additivity ensures that the risk measure is reasonable when adding two positions. It allows the decentralized calculation of the risk at an enterprise-wide level, since the sum of the risks of individual positions provides an upper bound on the enterprise-wide risk. Sub-additivity and homogeneity imply that the risk measure function is convex. This is consistent with risk aversion on the part of the user of these measures.

2.12.2 Value-at-Risk (VaR) and Conditional-Value-at-Risk (CVaR)

Value-at-risk is a percentile based metric that has become an industry standard for risk measurement purposes. It is usually defined as the maximal allowable loss with a certain confidence level $\alpha * 100\%$. Here we define *VaR* equivalently, in terms of returns, as the minimal portfolio return for a prespecified confidence level $\alpha * 100\%$. Thus,

$$VaR(\boldsymbol{x},a) = \min\{u: F(\boldsymbol{x},u) \geq 1-a\} = \min\{u: P\{R(\boldsymbol{x},\tilde{\boldsymbol{r}}) \leq u\} \geq 1-a\} \ \ (2.26)$$

$VaR(\boldsymbol{x}, \alpha)$ is the $(1-a) * 100\%$ percentile of the distribution of portfolio return.

Despite its popular use in risk measurement, VaR is not typically used in mathematical models for optimal portfolio selection. While its calculation for a certain portfolio $\boldsymbol{x}$ reveals that the portfolio return will be below $VaR(\boldsymbol{x}, \alpha)$ with a likelihood $(1-a) * 100\%$, it provides no information on the extent of the distribution's tail which may be quite long; in such cases, the portfolio return may take substantially lower values than VaR and the result is severe losses. VaR lacks a theoretical property for coherent risk measures, namely, sub-additivity. Moreover, VaR is difficult to optimize. When the asset returns are specified in terms of scenarios the VaR function is nonsmooth and non-convex with respect to the portfolio positions $\boldsymbol{x}$ and exhibits multiple local extrema. Efficient algorithms for solving problems with such objective functions are lacking.

<u>*Conditional value-at-risk*</u> $(CVaR)$ is a related risk measure. It is usually defined as the conditional expectation of losses exceeding VaR at a given confidence level. Here, we define $CVaR$ equivalently as the conditional expectation of portfolio returns below the VaR return. For continuous distributions, $CVaR$ is defined as

$$CVaR(\boldsymbol{x}, a) = \boldsymbol{\mathcal{E}}[R(\boldsymbol{x}, \tilde{\boldsymbol{r}}) | R(\boldsymbol{x}, \tilde{\boldsymbol{r}}) \leq VaR(\boldsymbol{x}, a)] \quad (2.27)$$

Hence, this definition of $CVaR$ that is applicable to continuous distributions measures the expected value of the $(1-a) * 100\%$ lowest returns for portfolio $\boldsymbol{x}$. For discrete distributions, the formula (2.25) gives a nonconvex function in portfolio positions $\boldsymbol{x}$, and is a non-subadditive risk measure. A definition of $CVaR$ for general distributions is:

$$CVaR(\boldsymbol{x}, a) = \left(1 - \frac{\sum_{\{s \in \Omega | R(x, r_s) \leq z\}} p_S}{1-a}\right) z + \frac{1}{1-a} \sum_{\{s \in \Omega | R(\boldsymbol{x}, \boldsymbol{r}_s) \leq z\}} p_S \, R(\boldsymbol{x}, \boldsymbol{r}_s) \quad (2.28)$$

where $z = VaR(\boldsymbol{x}, \alpha)$. As we consider discrete distributions, we will utilize this alternative definition of $CVaR$. Note that $CVaR$ as defined for discrete distributions in (2.26) may not be equal to the conditional expectation of portfolio returns below $VaR(\boldsymbol{x}, \alpha)$. This definition of $CVaR$ for discrete distributions measures only approximately the conditional portfolio returns below the respective $VaR(\boldsymbol{x}, \alpha)$ value. $CVaR$ quantifies the expected portfolio return in a low percentile of the distribution. Hence it can be used to exercise some control on the lower tail of the return distribution and thus, it is a suitable risk measure for skewed distributions. When the uncertain asset returns are represented by a discrete distribution $CVaR$ can be optimized by linear programming (LP). We follow for every scenario $s \in \Omega$ an auxiliary variable

$$y_s^+ = \max\,[0, z - R(\boldsymbol{x}, \boldsymbol{r}_s)],$$

which is equal to zero when the portfolio return for the particular scenario exceeds $VaR(\boldsymbol{x}, \alpha)$ and is equal to the return shortfall in relation to VaR when the portfolio return is below $VaR(\boldsymbol{x}, \alpha)$. Using these auxiliary variables we have:

$$\sum_{s\in\Omega} p_s y_s^+ = \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s y_s^+ + \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)> z\}} p_s y_s^+$$

$$= \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s (z - R(\boldsymbol{x},\boldsymbol{r}_s))$$

$$= z \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s - \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s\, R(\boldsymbol{x},\boldsymbol{r}_s)$$

$$= z(1-a) - \left(\left(1 - a - \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s \right) z + \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s\, R(\boldsymbol{x},\boldsymbol{r}_s) \right)$$

Dividing both sides of the equation by $(1-a)$ and rearranging terms we get equation (2.28)

$$z - \frac{\sum_{s\in\Omega} p_s y_s^+}{1-\alpha} = \left(1 - \frac{\sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s}{1-a} \right) z + \frac{1}{1-a} \sum_{\{s\in\Omega | R(\boldsymbol{x},\boldsymbol{r}_s)\le z\}} p_s\, R(\boldsymbol{x},\boldsymbol{r}_s)$$

From equations (2.26) to (2.28) we observe that the right hand side term of (2.28) is $CVaR(\boldsymbol{x}, \alpha)$. Therefore, the *conditional value-at-risk* of portfolio return can be optimized using a linear program with the left hand side expression of (2.28) as the objective function. The resulting LP that trades off the optimal $CVaR$-measure of portfolio return at a prespecified confidence level $a * 100\%$ against the expected portfolio return μ is written as

$$\begin{aligned} & Maximize \quad z - \frac{1}{1-a} \sum_{s=1}^{S} p_s y_s^+ \\ & s.t. \quad \boldsymbol{x} \in X, z \in \Re \\ & \boldsymbol{x}^{\mathsf{T}} \bar{\boldsymbol{r}} \ge \mu \qquad (2.29) \\ & y_s^+ \ge z - \boldsymbol{x}^{\mathsf{T}} \boldsymbol{r}_s \qquad s = 1,2,..,S \\ & y_s^+ \ge 0 \qquad s = 1,2,\dots,S \end{aligned}$$

Solving the parametric program (2.29) for different values of the expected portfolio return μ yields the $CVaR$-efficient frontier. For each expected return target μ , the optimal value of program (2.29) is the corresponding $CVaR(\boldsymbol{x},\alpha)$. The value of the free variable z at the optimal solution of (2.29) is the corresponding $VaR(\boldsymbol{x},\alpha)$ value. Program (2.29) optimizes the $CVaR$ risk measure for portfolio return and simultaneously determines the corresponding $VaR\ value\ (z)$. As defined in (2.26), in terms of portfolio return, $CVaR$ is a lower bound for $VaR(i.e., CVaR(\boldsymbol{x},\alpha) \leq VaR(\boldsymbol{x},\alpha))$. Hence, by maximizing $CVaR$ program (2.29) should be expected to yield larger values for VaR as well. Computational issues aside, there is an ongoing debate among academics and practitioners whether VaR or $CVaR$ is the most appropriate metric for risk management. VaR is the industry standard for risk measurement. On the other hand, $CVaR$ has achieved popularity as a suitable risk measure in the insurance industry and is gradually gaining acceptance in the financial community. Its appeal lies not only in its theoretical properties of coherence, but also in its ease of implementation in portfolio optimization models and its ability to reduce the tail of the distribution, thus exercising risk management control.

2.12.3 Mean Absolute Deviation (MAD)

In the mean absolute deviation framework risk is defined as the mean absolute deviation of portfolio return from its expected value:

$$MAD(\boldsymbol{x}) = \boldsymbol{\mathcal{E}}[|R(\boldsymbol{x},\tilde{\boldsymbol{r}}) - R(\boldsymbol{x},\bar{\boldsymbol{r}})|].$$

When the uncertain asset returns are represented in terms of a discrete scenario set the MAD metric becomes:

$$MAD(\boldsymbol{x}) = \sum_{\boldsymbol{s=1}}^{S} p_s |\ \boldsymbol{x}^{\mathsf{T}}\boldsymbol{r_s} - \boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{r}}|.$$

In this case MAD can be optimized by the following linear program:

$$\begin{aligned} & Minimize \quad \sum_{s=1}^{S} p_s y_s \\ & s.t. \quad \boldsymbol{x} \in X \\ & \boldsymbol{x}^{\mathsf{T}}\bar{\boldsymbol{r}} \geq \mu \\ & y_s \geq \boldsymbol{x}^{\mathsf{T}}(\boldsymbol{r_s} - \bar{\boldsymbol{r}}) \qquad s = 1,2, \dots, S \qquad\qquad (2.30) \\ & y_s \geq \boldsymbol{x}^{\mathsf{T}}(\bar{\boldsymbol{r}} - \boldsymbol{r_s}) \qquad s = 1,2, \dots, S \\ & y_s \geq 0 \qquad\qquad s = 1,2, \dots, S \end{aligned}$$

The auxiliary variables y_s are introduced to linearize the absolute value expression, akin to the approach followed earlier to linearize the piecewise linear function $max[0, z - R(\boldsymbol{x}, \boldsymbol{r_s})]$ in the $CVaR$ case. Again, by solving the parametric program (2.30) for various values of expected portfolio return μ we can construct the MAD-efficient frontier. MAD models have been applied to various portfolio optimization problems.

Chapter 3

Utility Theory

3.1 Introduction

In economics, utility is a measure of the relative fulfillment from the consumption of different goods and services. Given this measure, one may speak meaningfully of increasing or decreasing utility, and thereby explain economic behavior in terms of attempts to increase one’s utility. Our fundamental concern is the choice between timeless risky alternatives, which we call the theory of investor choice. The theory of investor choice is just a specific corner of what has come to be known as utility theory. The theory begins with the five main presumptions about the behavior of individuals when confronted with the undertaking of ranking risky alternatives and the supposition of nonsatiation. The theory ends by parameterizing the objects of decision as the mean and the variance of return and by mapping trade-offs between them that yield the same level of utility to investors. In general, decision making under uncertainty might be viewed as ranking alternative probability distributions of returns, in view of a consistent set of preferences. No specification is made about the return distribution and the empirical distribution is posited to represent the underlying distribution.

3.2 Five Axioms of Choice under Uncertainty

In order to develop the theory of rational decision making, it is important to make some exact presumptions about an individual’s behavior. These presumptions, known as the axioms of cardinal utility, provide the minimum set of requirements for consistent and rational behavior.

- *Axiom 1* → **Comparability**: For the whole set, S of uncertain outcomes, an individual can declare either that outcome x is preferred over outcome y $(x > y)$ or y is preferred to x $(y > x)$ or the individual is indifferent as to x and y $(x \sim y)$.

- *Axiom 2* → **Transitivity**: If an individual prefers x to y and y to z, then x is preferred to z. If $x > y$ and $y > z$, then $x > z$. If an individual is indifferent as to x and y and is also indifferent as to y and z, then he or she is indifferent as to x and z. If $x \sim y$ and $y \sim z$, then $y \sim z$.

- *Axiom 3* → **Strong Independence**: Suppose we construct a gamble where an individual has a probability a of receiving outcome x and a probability of $(1 - a)$ of receiving outcome z. We will write this gamble as $G(x, z : a)$. Strong independence says that if the individual is indifferent as to x and y, then he or she will also be indifferent as to a first gamble, set up between x and probability a and a mutually exclusive outcome z, and a second gamble, set up between y with probability a and the same mutually exclusive outcome, z. If $x \sim y$, then $G(x, z: a) \sim G(y, z: a)$

- *Axiom 4*→ **Measurability**: If outcome y is preferred less than x but more than z, then there is a unique a such that the individual will be indifferent between y and a gamble between x with probability a and z with probability $(1 - a)$.

  If $x > y > z \ or \ x \geq y > z$, then there exists a unique a, such that $y \sim G(x, z: a)$.

- *Axiom 5*→**Ranking**: If alternatives y and u both lie somewhere between x and z and we can establish gambles such that an individual is indifferent between y and a gamble x (with probability a_1) and z, while also indifferent between u and a second gamble, this time between x (with probability a_2) and z, then if a_1 is greater than a_2, y is preferred to u.

  If $x \geq y \geq z$ and $x \geq u \geq z$, then if $y \sim G(x, z: a_1)$ and $u \sim G(x, z: a_2)$, it follows that if $a_1 > a_2$, then $y > u$, or if $a_1 = a_2$, then $y \sim u$.

These axioms of cardinal utility illustrate the following assumptions about behavior. First all individuals are assumed to always make completely rational choices. Second, people are assumed to be able to make these rational choices among thousands of alternatives.

3.3 Existence of Utility Function

The relationship between wealth and the utility of consuming this wealth is described by a utility function. Each investor will have a different utility function. The utility function will have two properties. First, it will be order preserving. In other words, if we measure the utility of x as greater than the utility of y then $U(x) > U(y)$, which means that x is actually preferred to y.
This mathematically means that : $U[G(x, y: a)] = aU(x) + (1 - a)U(y)$.
The second property is that the expected utility can be used to rank combinations of risky alternatives. Because different outcomes come with an associated probability distribution, different investments will be indexed by their expected utility.
Generally, we can write the expected utility of wealth as follows:

$$E[U(W)] = \sum_i p_i U(W_i) \qquad (3.1)$$

Given the five axioms of rational investor behavior and the additional presumption that all investors always prefer more wealth to less, we can conclude that investors will always seek to maximize their expected utility of wealth. All investors will use it as their objective function. In other words, they will seem to calculate the expected utility of wealth for all possible alternative choices and then choose the outcome that maximizes their expected utility of wealth. An important thing to keep in mind is that utility functions are specific to individuals. There is no way to compare one individual's utility function to another's. Another important property of cardinal utility functions is that we can sensibly talk about increasing or decreasing marginal utility. Consider two risky alternatives with outcomes x and y respectively. The difference between these two outcomes is marginal utility. Mathematically this can be expressed as follows:

$$\frac{U(x) - U(y)}{\psi(x) - \psi(y)} = constant, \qquad (3.2)$$

where $U(.)$ and $\psi(.)$ are the two utility functions. Changes in utility between any two levels of wealth have the exact same meaning on the two utility functions; that is, one utility function is just a "transformation" of the other.

3.4 Properties of Utilities Functions

Many investors do not obey all the rationality postulates when faced with a series of choice situations, even though they may find the underlying principles perfectly reasonable. Investors when faced with more complicated choice decisions, encounter aspects of the problem that were not of concern to them in the simple choice situations. The first constraint placed on a utility function is that it be consistent with more being preferred to less. Standard utility functions are monotone increasing and represent non-satiated preferences. Individuals prefer more wealth to less, so the first derivative of the utility function, with respect to wealth is positive, $U'(W) > 0$. This attribute, aka nonsatiation, simply signifies that the utility of more dollars $(W + 1)$ is higher than the utility of less (W) dollars. Equivalently, more wealth is always preferred to less wealth, in other words, the marginal utility of wealth is positive.

The second constraint concerns the preferences for risk. Risk aversion, risk neutrality and risk-seeking behaviors are all defined relative to a fair gamble. A fair gamble is a gamble priced at its expected value; a risk-averse investor will always reject a fair gamble in favor of its mean value. A person who prefers the fair gamble is a risk lover; one who is indifferent is risk neutral; and one who prefers the actuarial value with certainty is a risk averter. The actuarial value of the gamble is its average outcome. Individuals exhibit risk aversion, hence this attribute implies a utility function that is convex or concave downwards, increasing thus at a decreasing rate. Mathematically this can be expressed as, $U''(W) < 0$.

Because we gain more utility from the actuarial value of the gamble obtained with certainty than from taking the gamble itself, we are risk averse. In general, if the utility of expected wealth is greater than the expected utility of wealth, the individuals will be risk averse. The three definitions are the following:

$$If\ U[E(W)] > E[U(W)] \text{ ,then we have risk aversion} \Rightarrow U''(W) < 0$$

$$If\ U[E(W)] = E[U(W)] \text{,then we have risk neutrality} \Rightarrow U''(W) = 0$$

$$If\ U[E(W)] < E[U(W)] \text{,then we have risk loving} \Rightarrow U''(W) > 0$$

We must denote that risk neutrality implies a linear utility function; risk loving implies a convex utility function and finally if the utility function is strictly concave, then we will be risk averse. *Figure 3.1* depicts the three utility functions in the wealth space exhibiting alternative properties with respect to risk.

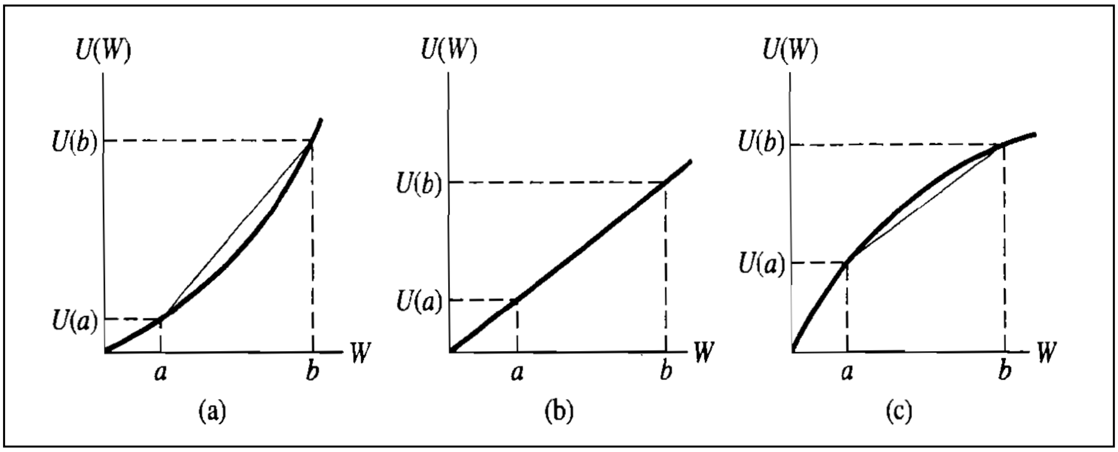

Figure 3.1: Three utility functions with positive marginal utility: (a) risk-lover; (b) risk neutral; (c) risk averter.

It is even possible to calculate the maximum amount of wealth an individual would be willing to give up in order to avoid the gamble. This is called the *risk premium*. We have the ability to compute the risk premium as the difference between an individual's expected wealth, given the gamble, and the level of wealth that the individual would accept with certainty if the gamble were removed, that is what we called *the certainty equivalent wealth*. Moreover, there is another norm that might be called *the cost of gamble*. It is defined as the difference between an individual's current level of wealth and his or her certainty equivalent wealth. For a risk averter *the risk premium* as defined above is always positive, whereas *the cost of the gamble* can be positive, negative or zero, depending on the risk of the gamble and how much it is expected to change one's current level of wealth.

The third property of utility functions that is sometimes presumed is an assumption about how the investor's preferences change with a change in wealth. If the investor increases the amount invested in the risky assets as wealth increases, then the investor is said to exhibit *decreasing absolute risk aversion*. If the investor's investment in risky assets is unchanged as wealth changes, then he or she is said to exhibit constant absolute risk aversion. Finally, if the individual invests fewer dollars in risky assets as wealth increases, then he is said to exhibit increasing absolute risk aversion. The index that can be used to measure an investor's absolute risk aversion *(ARA)* is the *Pratt-Arrow measure* of a local risk premium. We shall define the measure of absolute risk aversion as:

$$A(W) = -\frac{U''(W)}{U'(W)} \qquad (3.3)$$

It is called absolute risk aversion because it measures risk aversion for a given level of wealth. The Pratt-Arrow definition of risk aversion is quite important because it provides much more insight into people's behavior when confronted with risk. The table below summarizes the relevant properties.

Condition	Definition	Property of $A(W)$
Increasing Absolute Risk Aversion (IARA)	As wealth increases hold fewer dollars in risky assets	$A'(W) > 0$
Constant Absolute Risk Aversion (CARA)	As wealth increases hold the same amount of dollars in risky assets	$A'(W) = 0$
Decreasing Absolute Risk Aversion (DARA)	As wealth increases hold more dollars in risky assets	$A'(W) < 0$. The utility function is positively skewed; that is $U'''(W) > 0$

Table 3.1: Properties and alternative conditions concerning ARA.

The final characteristic that is used to restrict the investor's utility function is how the percentage of wealth invested in risky assets changes as wealth changes. We can multiply the measure of absolute risk aversion by the level of wealth to obtain what is known as relative risk aversion *(RRA)*:

$$R(W) = -W\frac{U''(W)}{U'(W)} = WA(W) \qquad (3.4)$$

Relative risk aversion is related to absolute risk aversion but *RRA* concerns the change in the percentage investment in risky assets as wealth changes, whereas *ARA* refers to dollar amounts invested in risky assets. The following table presents all the relevant properties for *relevant risk aversion*.

Condition	Definition	Property of $R'(W)$
Increasing Relative Risk Aversion (IRRA)	Percentage invested in risky assets declines as wealth increases.	$R'(W) > 0$
Constant Relative Risk Aversion (CRRA)	Percentage invested in risky assets is unchanged as wealth increases.	$R'(W) = 0$
Decreasing Relative Risk Aversion (DRRA)	Percentage invested in risky assets increases as wealth increases.	$R'(W) < 0$

Table 3.2: Properties and alternative conditions concerning RRA.

3.5 Quadratic Utility Functions and their Limitations

We can use the previous definitions of risk aversion to provide a more detailed examination of different types of utility functions to see whether or not they have decreasing ARA and constant RRA. The quadratic utility function has been widely used in academic literature. This type of utility function may be satiated and entail problematically increasing absolute and relative risk aversion coefficients. While there is a general agreement that most investors exhibit decreasing absolute risk aversion, there is much less agreement considering relative risk aversion. Moreover, people often assume constant relative risk aversion. The justification for this, however, is often one of convenience rather than a belief about descriptive accuracy.

We can now introduce the properties of the quadratic utility function that can be written as follows:

$$Quadratic\ Utility\ Function: U(W) = aW - bW^2;$$
$$First\ Derivative, Marginal\ Utility: U'(W) = a - 2bW;$$
$$Second\ Derivative\ with\ respect\ to\ change\ in\ wealth: U''(W) = -2b.$$

For the quadratic utility function, ARA and RRA are:

$$A(W) = -\frac{2b}{a - 2bW}, A'(W) > 0 \qquad (3.5)$$
$$R(W) = \frac{2b}{\left(\frac{a}{W}\right) - 2b}, R'(W) > 0 \qquad (3.6)$$

As we can observe, unfortunately, the quadratic utility function exhibits increasing ARA and increasing RRA.

3.6 Power Utility Functions and their Properties

In economics, the isoelastic function for utility, also known as power utility function is used to express utility in terms of consumption or some other economic variable that a decision-maker is concerned with such as his or her level of wealth. The power utility function is a special case of *Hyperbolic Absolute Risk Aversion* and at the same time is the only class of utility functions with *Constant Relative Risk Aversion*, which is why it is also called the $CRRA$ utility function. It can be written as follows:

$$U(W) = -W^{-1}, \quad U'(W) = W^{-2} > 0, \;\; U''(W) = -2W^{-3} < 0.$$

For this power utility function, ARA and RRA are:

$$A(W) = -\frac{2W^{-3}}{W^{-2}} = \frac{2}{W}, \qquad A'(W) < 0, \quad (3.7)$$

$$R(W) = W\frac{2}{W} = 2, \;\; R'(W) = 0 \qquad (3.8)$$

We can observe that this particular type of utility function exhibits all the reasonable properties: the marginal utility of wealth is positive, it decreases with increasing wealth, the measure of ARA decreases with increasing wealth and RRA is constant.

Stochastic Dominance and Decision Rules

3.7 Stochastic Dominance and Decision Rules

Stochastic Dominance is presently employed in various areas of economics, finance and statistics. The concept of Stochastic Dominance is quite old. For equal mean distributions, Karamata proved in 1932 a theorem which is very similar to the Second Degree Stochastic Dominance. Blackwell (1951,1953) used similar concepts in the comparison of the statistical experiments. The application of these Stochastic Dominance concepts in decision theory began about 40 years ago. Statisticians employ majorization theory of various orders, which is parallel to Stochastic Dominance theory. However, the application of majorization theory in statistics is quite remote from the analysis and applications of Stochastic Dominance theory in economics and finance. The theory of Stochastic Dominance and its many theoretical and empirical extensions in economics and finance were developed only after 1969-1970, when four papers were independently published by Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970) and Whitmore (1970). Since that time, hundreds of papers have been written on the topic.
Since the publication of these original four papers, the following four main areas have developed:

- ✓ *Further theoretical development*. These theoretical studies focus on the following topics: *(i)* the ordering of uncertain options for specific return distributions, *(ii)* Stochastic Dominance rules for restricted classes of utility functions, *(iii)* Stochastic Dominance rules when a riskless asset is allowed, *(iv)* Stochastic Dominance for the multiperiod case, *(v)* diversification between risky assets, *(vi)* multivariate analysis, *(vii)* the role of Stochastic Dominance rules in nonlinear utility theory, and *(viii)* Stochastic Dominance of transformed random variables.
- ✓ *Application of Stochastic Dominance rules to empirical data*. Various algorithms have been developed in this area. Several necessary conditions have been proposed which increase the efficiency of calculations.
- ✓ *Applications of Stochastic Dominance rules to other economic and financial issues*. These issues include *(i)* optimum financial leverage with bankruptcy, *(ii)* optimum production, *(iii)* measuring income inequality, *(iv)* the analysis and definition of risk, *(v)* measuring bankruptcy risk by using data taken from bond market, *(vi)* portfolio insurance, *(vii)* option valuation.

- ✓ *Application of Stochastic Dominance rules in statistics*. While majorization theory is widely used in statistics and in experimental design, a relatively new application involves choosing among various estimators of a given parameter. The Stochastic Dominance rules enable the selection of Stochastic Dominance efficient estimators.

3.7.1 Partial Ordering: Efficient and Inefficient Sets

So far we have discussed the axioms of investor preference, then used them to develop utility functions and finally employed the utility functions to measure risk premium and derive measures of risk aversion. Obviously, any decision-maker, whether risk averse or not, will seek to maximize the expected utility of his or her level of wealth. If there is full information on preferences, we simply compute the expected utility of all the competing investments and select the one with the highest expected utility. In such case, we reach at a complete ordering of the investments under consideration. Moreover, with a complete ordering, we can rank the investments from best to worst. In general, however, we have only partial information on preferences (e.g., risk aversion) and, therefore, we arrive only at a partial ordering of the available investments. Stochastic Dominance rules as well as other investments rules such as the mean-variance rule employ partial information on the investor's preferences or the random variables and therefore, they produce only partial ordering.

Suppose that all we know is that the utility function is non-decreasing with $U' \geq 0$, namely, investors always prefer more money than less money. As a consequence, we have partial information on the utility function and its exact shape is not known. We will now introduce some definitions, all of which are commonly used in the financial literature and which are needed for the explanation of partial and complete ordering.

The feasible set is defined as the set of all available investments under consideration. Then, using an investment rule, we can divide the whole feasible set, into two sets: the efficient set and the inefficient set. These two sets are mutually exclusive and comprehensive; namely: $(Efficient\ Set) U\ (Inefficient\ Set) = Feasible\ Set$

Figure 3.2 demonstrates the division of the feasible set, into two sets, the inefficient set and the efficient set. In this figure, the feasible set includes five investments A, B, C, D and E. Each investment included in the feasible set must be either in the efficient set or in the inefficient set. Now let's assume that the only information we have is that $U' \geq 0$. Thus, $U \in \boldsymbol{U_1}, if\ U' \geq 0$ where $\boldsymbol{U_1}$ is the set of all non-decreasing utility functions. We demonstrate below the concept of the efficient set and the inefficient set and the relationship between the two sets for this particular type of information (namely $U \in \boldsymbol{U_1}$). Before we define the efficient set and the inefficient set formally, we need the following definitions.

Dominance in $\boldsymbol{U_1}$**:** We say that investment A dominates investment B in $\boldsymbol{U_1}$ if for all utility functions such that $U \in \boldsymbol{U_1}$**,** $E_A U(x) \geq E_B U(x)$ and for at least one utility function $U_0 \in \boldsymbol{U_1}$ there is a strict inequality.

Efficient Set in $\boldsymbol{U_1}$: An investment is included in the efficient set if there is no other investment that dominates it. The efficient set includes all the undominated investments. Referring to *Figure 3.2* we can say that investments A and B are efficient. Neither A nor B dominates each other. In particular, there is a utility function $U_1 \in \boldsymbol{U_1}$ such that $E_A U_1(x) > E_B U_1(x)$, and there is another utility function, $U_2 \in \boldsymbol{U_1}$ such that $E_B U_2(x) > E_A U_2(x)$. Therefore, neither A nor B is the ''best'' for all investors included in the group $U \in \boldsymbol{U_1}$. Some investors may choose A and some may prefer B, and there is no dominance between A and B.

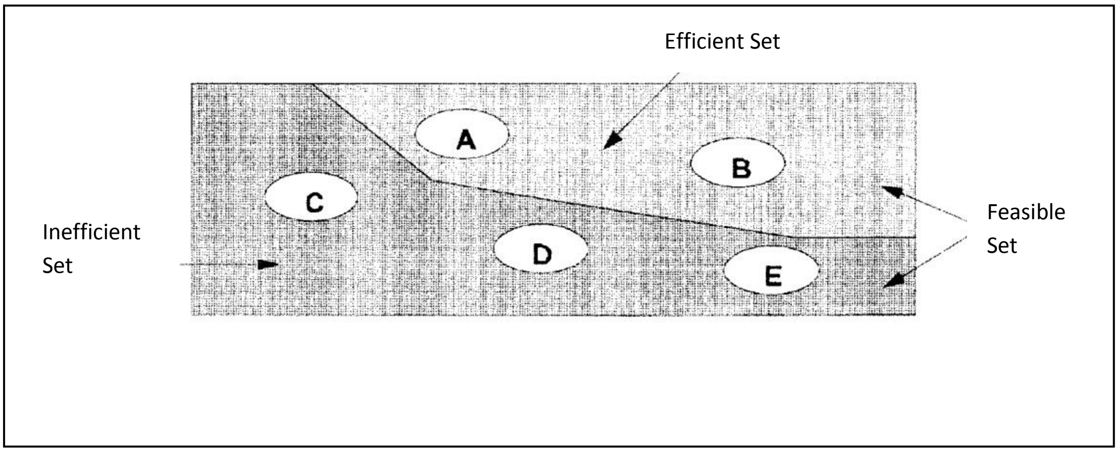

Figure 3.2: The feasible, the Efficient and the Inefficient Sets.

Inefficient Set in $\boldsymbol{U_1}$: The inefficient set includes all inefficient investments. The definition of an inefficient investment is that there is at least one investment in the efficient set that dominates it. *Figure 3.2* shows that investments C, D and E are inefficient. For example, we might have the following relationships:

$$E_A U(x) \geq E_C U(x)$$

$$E_A U(x) \geq E_D U(x)$$

$$E_B U(x) \geq E_E U(x)$$

$$for\ all\ U \in \boldsymbol{U_1}$$

Thus, the efficient investment A dominates investment C and D, and the efficient investment B dominates investment E. There is no need for an inefficient investment to be dominated by all efficient investments. One dominance is enough to relegate an investment to the inefficient set. The partition of the feasible set to the efficient set and inefficient set depends on the information available. In the previous example, we have assumed that $\in \boldsymbol{U_1}$. If, for example in addition to $U' > 0$,we assume also that $U'' < 0$ or any other relevant restriction, we will get another partition of the feasible set to inefficient set and efficient set reflecting the additional information. In general, for any given piece of information, the smaller the efficient set relative to the feasible set, the better off the investors. In investment choice with partial information there are two decision stages; the first involving the investment consultant and the second the individual investor. The two stages are as follows:

- **The objective decision:** In the first stage, the initial screening of investments is accomplished by partitioning the feasible set into the efficient set and the inefficient set. Due to the fact that the efficient set generally includes more than one investment and we cannot tell which one is the best, this stage provides only partial ordering. If we posses full information such as the specific type of the utility function, then the efficient set will include only one investment and we arrive at a complete ordering of the investments.
- **The subjective decision:** The optimum investment choice by the investor from the efficient set. This optimal choice maximizes the investor's expected utility. This is a subjective decision because it depends on the investor's preferences. All investors will choose their optimal portfolio from the efficient set according to their preferences. Therefore, there will be little or no agreement between investors, because each one of them will select his optimal portfolio according to his particular preferences.

3.7.2 First Degree Stochastic Dominance (FSD)

In this section, we present the First Stochastic Dominance in detail. However, because Stochastic Dominance rules rely on distribution functions, some discussion of probability function, density function and cumulative probability function are called for before turning to the FSD rule.

3.7.2.1 Probability function, density function and cumulative probability function

Consider the pair (x, p) where x is the outcome and $p(x)$ is its corresponding probability that is called a probability function. If the random variable x is continuous, then the probability function is replaced by the density function $f(x)$. The cumulative probability function denoted by $F(x)$ is given as:

$$F(x) = P(X \leq x) = \sum_{X \leq x} P(x) \quad for\ a\ discrete\ distribution \qquad (3.9)$$

and

$$F(x) = \int_{-\infty}^{x} f(t)dt \quad for\ a\ continuous\ random\ variable \qquad (3.10)$$

where X denotes a random variable and x a particular value.

3.7.2.2 The First Stochastic Dominance Rule

Suppose, now that an investor wishes to rank two investments whose cumulative distributions are F and G, respectively. The FSD rule is a criterion that tells us whether one investment dominates another investment where the only available information is that $U \in U_1$, that is $U' \geq 0$. In fact, this is the weakest assumption on preference because we assume only that investors like more money rather than less money, which conforms to the monotonicity axiom. Moreover, we assume that U is a continuous non-decreasing function which implies that it is differentiable apart from a set of points whose measure is zero. In what follows we presume investors maximize the *von Neumann-Morgenstern expected utility*.

$Theorem$ 1: Let F and G be the cumulative distributions of two distinct uncertain investments. Then F dominates G by FSD, which we denote by FD_1G where D_1 denotes dominance by the first degree and the subscript 1 indicates that we assume only one piece of information on U, namely that U is nondecreasing, for all $U \in U_1$ if and only if $F(x) \leq G(x)$ for all values of x, and there is at least some x_0 for which a strong inequality holds. As FSD relates to $U \in U_1$, it can be expressed as follows:

$$F(x) \leq G(x)\ for\ all\ x \quad \Leftrightarrow \quad E_F U(x) \geq E_G U(x) \quad for\ all\ U \in U_1 \qquad (3.11)$$

In *Figure 3.3* we can observe five cumulative distributions representing the feasible set that include all the possible investments. It is easy to show that the FSD efficient set contains F_3 and F_4 and the FSD inefficient set contains F_1, F_2 and F_5. Several conclusions can be drawn from *Figure 3.3*:

i. FSD dominance requires that the two distributions being compared do not cross but they can tangent each other. For example, F_3 dominates F_2 in spite of the fact that there is a range where $F_2(x) = F_3(x)$. F_3 dominates F_2 because the following holds: $F_3(x) \leq F_2(x)$ for all values and there is at least one value $x_0$for which $F_3(x_0) < F_2(x_0)$.

ii. An inefficient investment should not be dominated by all efficient investments. Dominance by one investment is enough. F_4 does not dominate F_1, F_2 and F_5 because they intersect but F_3 dominates all these three investments. Therefore, in order to be relegated into the inefficient set, it is sufficient to have one investment that dominates the inefficient investment.

iii. In the inefficient set, one investment may or may not dominate another investment in the inefficient set. $F_1 D_1 F_5$ but $\not{F}_1 D_1 F_2$ and $\not{F}_2 D_1 F_1$. However, dominance or no dominance within the inefficient set is irrelevant because all investments included in this set are inferior; no investor with preference $U \in \boldsymbol{U_1}$ will select an investment from the inefficient set.

iv. An investment within the inefficient set cannot dominate an investment portfolio within the efficient set because if so dominance were to exist then the latter would not be included in the efficient set. For instance, if F_2 were to dominate F_3, then F_3would not be an efficient investment.

v. Eventually, all investments within the FSD efficient set must intercept. In our example, F_3 and F_4 intercept. Without such an interception, one distribution would dominate the other and neither would be efficient.

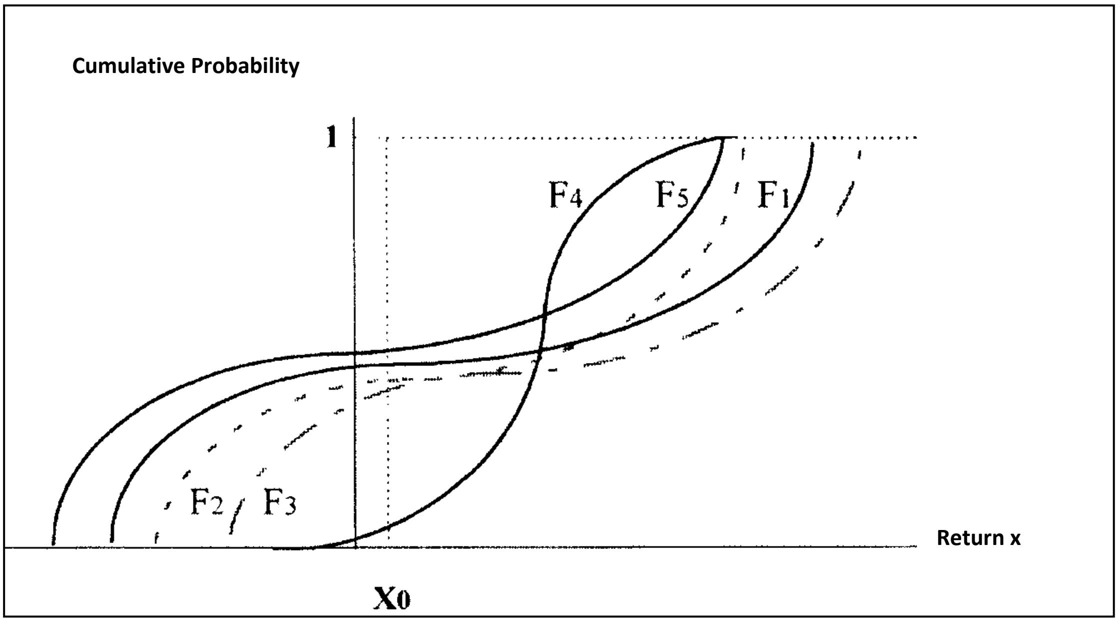

Figure 3.3: The FSD Efficient and Inefficient Sets.

The interception of $F_3$and F_4 implies that there is a $U_1 \in \boldsymbol{U_1}$ such that:

$$E_{F_3}U_1(x) > E_{F_4}U_1(x)$$

and there is another utility function $U_2 \in \boldsymbol{U_1}$ such that :

$$E_{F_4}U_2(x) > E_{F_3}U_2(x)$$

Thus, all investors in the class $U \in \boldsymbol{U_1}$ will agree on the content of the FSD efficient and inefficient sets; none of them will select their optimum choice from the inefficient set. However, they may disagree on the selection of the optimal investment from the efficient set; one may choose F_3 and another may choose F_4.

3.7.2.3 Optimal Rule, Sufficient Rules and Necessary Rules for FSD

An optimal decision rule is defined as a decision rule, which is necessary and sufficient for dominance. The FSD rule is the optimal rule for $U \in \boldsymbol{U_1}$ because it is a sufficient and a necessary condition for FSD. Mathematically, an optimal rule for the set of $U \in \boldsymbol{U_1}$ is defined as follows:

$$E_F U(x) \geq E_G U(x) \;\; for \;\; all\, U \in U_1 \;\Leftrightarrow FD_1G \qquad (3.12)$$

Especially, FD_1G implies that for every $U \in \boldsymbol{U_1}$, F is preferred over G and the converse also holds; if it is known that for every $U \in \boldsymbol{U_1}$, F is preferred over G, then $F(x) \leq G(x)$ holds for all values of x with a strict inequality for some x_0.
An optimal rule is the best available rule for a given set of information. Suppose that we know that $U \in \boldsymbol{U_1}$ but there is no information on the precise slope of the utility function. This means that there is no better rule than the FSD for the assumption asserting that $U \in \boldsymbol{U_1}$ which, in turn, implies that there is no other investment rule that provides a smaller efficient set than the FSD efficient set. Hence, an optimal decision rule for all U such that $U \in \boldsymbol{U_1}$ provides the smallest efficient set for the given information on preferences.
Let's assume that there is a sufficient rule for U_1 which we denote by s. If F dominates G by this sufficient rule denoted as FD_sG then $E_F U(x) \geq E_G U(x)$ for all $U \in \boldsymbol{U_1}$. Any decision rule with the previous property is defined as a sufficient investment rule. At this point, we will demonstrate a few sufficient rules for $U \in \boldsymbol{U_1}$.

- <u>Sufficient Rule 1</u>: F dominates G if $Min_F(x) \geq Max_G(x)$.
  This is a sufficient rule because whenever it holds, FD_1G, which in turn implies that $E_F U(x) \geq E_G U(x)$ for all $U \in U_1$.

- Sufficient Rule 2: F dominates G if $F(x) \leq G(x)$ for all x and there is at least one value x_0 such that: $F(x_0) + a \leq G(x_0)$, where a is some fixed positive number.

Suppose now that $E_F U(x) \geq E_G U(x)$ for all $U \in \boldsymbol{U_1}$, implies that some condition must hold such that $E_F(x) \geq E_G(x)$. Then we call this condition a necessary rule for dominance. We refer below the three more necessary rules for dominance in $\boldsymbol{U_1}$.

- Necessary Rule 1 → *The means*.
  If FD_1G, then the expected value of F must be greater than the expected value of G. Thus, $E_F(x) \geq E_G(x)$ is a necessary condition for FSD.

- Necessary Rule 2 → *The Geometric Means*.
  If FD_1G, then the geometric mean of F must be larger than the geometric mean of G.

- Necessary Rule 3 → *The ''Left Tail'' condition.*
  If FD_1G, it is necessary that: $Min_F(x) \geq Min_G(x)$.
  This means that distribution G starts to accumulate area before distribution F. This is called the ''left tail'' condition because the cumulative distributions imply that G has a thicker left tail.

3.7.3 Second Degree Stochastic Dominance (SSD)

So far, the only assumption that we have made is that $U \in \boldsymbol{U_1}$. There is much evidence that most, if not all, investors are probably risk averters. Hence, we develop a decision rule that is appropriate for all risk averters. In the subsequent analysis, we will deal only with non-decreasing utility function, $U \in \boldsymbol{U_1}$, and we have added the assumption of risk aversion. As we have already mentioned in previous sections, risk aversion can be defined in the following alternative ways:

a) The utility function has a non-negative first derivative and a non-positive second derivative and there is at least one point at which: $U' > 0$ and one point at which $U'' < 0$.
b) If we take any two points on the utility function and connect them by a chord, then the chord must be located either below, or on, the utility function and there must be at least one chord which is located strictly below the utility function.

c) The expected utility is smaller or equal to the utility of the expected return. The property of concave functions, that is $U' \geq 0$ and $U'' \leq 0$ is called *Jensen's Inequality*: accordingly, for any concave function, the following will hold:

$U[E(W)] \geq E[U(W)]$, where W is the level of wealth.

d) A risk averter will not play a fair game. A fair game is defined as a game in which the price of a ticket to play the game is equal to the expected price.

e) Risk averters will be ready to pay a positive risk premium in order to insure their wealth.

We can define the set of all concave utility functions corresponding to risk aversions by U_2. It is obvious, that $\boldsymbol{U_2} \subseteq \boldsymbol{U_1}$, where $\boldsymbol{U_1}$ corresponds to FSD. In the next theorem, we provide a decision rule for all $U \in \boldsymbol{U_2}$**.** In the first place, we assume continuous random variables and then we extend the results to discrete random variables.

$Theorem\ 2$: Let F and G be two investments whose density functions are $f(x)$ and $g(x)$, respectively. Then F dominates G by second degree stochastic dominance (SSD), which we denote by FD_2G for all risk averters if and only if:

$$\int_{-\infty}^{x} [G(t) - F(t)]dt \geq 0 \ \ for\ all\ values\ of\ x, \qquad (3.13)$$

and there is at least one x_0 for which a strong inequality holds. This Theorem can also be stated as follows:

$$\int_{-\infty}^{x} [G(t) - F(t)]dt \geq 0 \ \ for\ all\ \ x \ \Leftrightarrow \quad E_F U(x) \geq E_G U(x)$$
$$for\ all\ U \in U_2 \qquad (3.14)$$

The SSD integral condition for dominance implies that the area enclosed between the two distributions under consideration should be non-negative up to every point x. When we examine whether F dominates G, whenever F is below G, we denote the area enclosed between the two distributions by ''+'' area, and whenever G is below F, we denote the area enclosed between the two distributions by "-" area. When we examine whether G dominates F, the opposite area signs are used. *Figure 3.4* illustrates two cumulative distributions F and G. SSD dominance may exist irrespective of the number of times that the two distributions intersect. As we can observe F dominates D by SSD. The integral $\int_{-\infty}^{x}[G(t) - F(t)]dt > 0$ for all values of x, and there is at least one strong inequality, say at x_1. Therefore F dominates G if for any negative area, there is a positive area located to the left of x_2 which is equal or larger than the negative area.

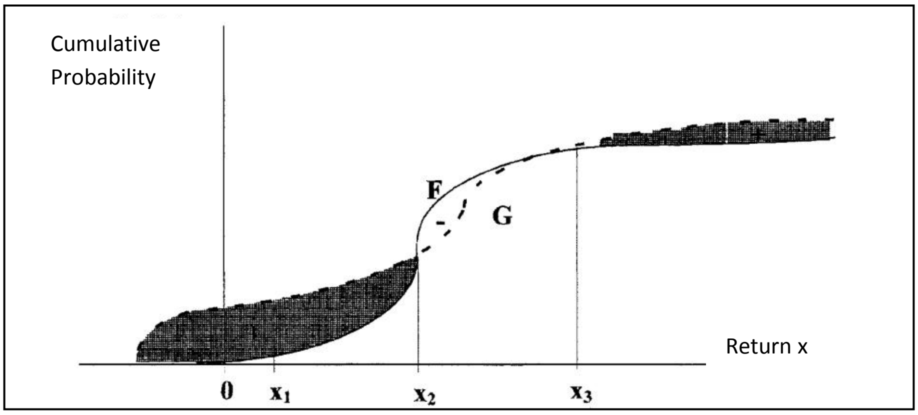

Figure 3.4: The area enclosed between the two distributions F and G. F dominates G by SSD.

In *Figure 3.4* there are only a few intersections of F and G. Let us generalize the SSD condition for a larger number of intersections between F and G. By the integral condition for any negative area there must be positive areas located earlier such that the sum of the positive areas is larger than the sum of all negative areas accumulated up to x_3. We denote by S^- and S^+ the negative and the positive areas, respectively. We then employ the absolute values of the areas for all area comparisons. Suppose that F and G intersect n times, $n = 1, 2, \ldots$ We order all the areas enclosed between F and G from the lowest intersection points of F and G to the highest intersection points of F and G as follows:

$$S_1, S_2, S_3, \ldots \ldots . S_n$$

where S_i can be a positive area or a negative area. Now assume that S_i^- is the first negative area. Then by the SSD rule, for the first negative area S_i^- , we must have that:

$$S_i^- \leq \sum_{j=1}^{i-1} S_j^+ \qquad (3.15)$$

Now suppose that i is the first negative area and m is the second negative area. Then by the SSD, we must have that:

$$S_i^- + S_m^- \leq \sum_{j=1}^{m-1} S_j^+ \qquad (3.16)$$

Generally, for the k^{th} negative area, we must have that:

$$\sum_{i=1}^{k} S_i^- \leq \sum_{j=1}^{l} S_i^+ \qquad (3.17)$$

where there are l positive areas before the k^{th} negative area. Namely, up to any point corresponding to a negative area, the sum of the positive areas must be larger than the sum of the negative areas.

3.7.3.1 Sufficient Rules and Necessary Rules for SSD

As in the case of FSD, there exist many sufficient rules for risk aversion which imply SSD. We will consider here three of such sufficient rules.

- Sufficient Rule 1: The FSD rule is a sufficient rule for SSD. Recall that if FD_1G then $F(x) \leq G(x)$ for all x. Therefore $G(x) - F(x) \geq 0$ for all x and because the integral of non-negative numbers is non-negative, we have :

  $$F(x) \leq G(x) \Rightarrow \int_{-\infty}^{x}[G(t) - F(t)]dt > 0 \Rightarrow E_F U(x) \geq E_G U(x)\ for\ all\ U \in U_2.$$

  Thus, if risk aversion is assumed, the FSD rule can be employed and any investment relegated to the inefficient set with FSD will also be relegated to the inefficient set with SSD. However, this sufficient rule may result in a relatively large efficient set.

- Sufficient Rule 2: $Min_F(x) \geq Max_G(x)$ is a sufficient rule for SSD. Like the FSD rule, this rule implies that $F(x) \leq G(x)$ for all values of x and because the latter implies SSD, we can state that:

  $$Min_F(x) \geq Max_G(x) \Rightarrow FSD \Rightarrow SSD \Rightarrow E_F U(x) \geq E_G U(x)\ for\ all\ U \in U_2.$$

  Indeed, any rule, which is sufficient for the FSD, must be also be sufficient for SSD because FSD dominance implies SSD dominance.

- Sufficient Rule 3: This sufficiency rule for SSD bears no relationship to FSD dominance; namely F dominates G if:

  $$\int_{-\infty}^{x}[G(t) - F(t)]dt \geq k, for\ all\ values\ of\ x\ where\ k > 0.$$

  This "$k\, rule$" is sufficient for SSD because if it holds for all x, then $\int_{-\infty}^{x}[G(t) - F(t)]dt \geq 0$.

We now turn to the necessary rules for risk aversion that imply SSD.

- Necessary Rule 1 → *The means*. $E_F(x) \geq E_G(x)$ is a necessary condition for F over G in U_2. Note that unlike FSD here a strong inequality $E_F(x) > E_G(x)$ is not a necessary condition for SSD.

- Necessary Rule 2 → *The Geometric Means*.
  $\bar{x}_{\text{geo}}(F) \geq \bar{x}_{\text{geo}}(G)$ is a necessary condition for dominance of F over G by SSD.

- Necessary Rule 3 → *The ''Left Tail'' condition.*
  A necessary rule for FD_2G is that $Min_F(x) \geq Min_G(x)$, namely the left tail of G must be "thicker".

3.7.4 *Third Degree Stochastic Dominance (TSD)*

3.7.4.1 *The definition of positive skewness as a motivation for TSD*

So far we have assumed either that $U' \geq 0$ from which we derived the corresponding FSD rule or alternatively that $U' \geq 0$ and $U'' \leq 0$ from which we derived the corresponding SSD rule. In this section we derive a decision rule called Third Degree Stochastic Dominance corresponding to the set of utility functions $U \in \boldsymbol{U_3}$ where $U' \geq 0, U'' \leq 0$ and $U''' \geq 0$. However before we turn to this decision rule, we must first discuss the economic rationale for the additional assumption asserting that $U''' \geq 0$. As we have already mentioned in previous sections the assumptions $U' \geq 0$ and $U'' \leq 0$ assume that the investor prefers more money to less money and that the investor is risk averter, respectively. The economic justification for $U''' \geq 0$ is related to the distribution skewness.
Skewness of a distribution of rate of return or the distribution's third central moment, denoted by μ_3 is defined as follows:

$$\mu_3 = \sum_{i=1}^{n} p_i (x_i - Ex)^3 \qquad (3.18)$$

for discrete distributions, where n is the number of observations and (p_i, x_i) is the probability function, and

$$\mu_3 = \int_{-\infty}^{\infty} f(x)(x - Ex)^3 dx \qquad (3.19)$$

for continuous distributions.

The prizes of a lottery game are generally positively skewed because there is a small probability of winning a very large prize. Additionally, the value of an uninsured house is negatively skewed because of the small probability of a heavy loss due to a fire. Finally, for symmetrical distributions such as the normal distribution the negative and positive deviations cancel each other out and the skewness is zero. In particular, investors choose to insure their houses, in order to reduce the variance of the future value as well as the negative skewness of an uninsured house drops to zero. This behavior of purchasing home insurance can be interpreted in two ways: People insure their houses because they dislike variance and they also dislike negative skewness. Home insurance ensures a certain income; thus the insurance company is actually selling the negative skewness as well as the variance. Similarly, when people buy a lottery ticket, variance and positive skewness are created. Therefore, buying a lottery ticket can be explained by asserting that the investor likes variance or the investor likes positive skewness because with a lottery ticket both variance and skewness increase.

In order to explain the relationship between U''' and skewness more precisely, we expand the utility function into a Taylor series around the point $(w + Ex)$ where the utility function is $U(w + x)$, w denotes the initial level of wealth and x is a random variable.

$$U(w+x) = U(w+Ex) + U'(w+Ex)(x-Ex) + \frac{U''(w+Ex)}{2!}(x-Ex)^2 + \frac{U'''(w+Ex)}{3!}(x-Ex)^3 + \ldots\ldots$$

Taking the expected value from both sides and using the fact that $E(x - Ex) = 0$ yields:

$$EU(w+x) = U(w+Ex) + \frac{U''(w+Ex)}{2!}\sigma_\chi^2 + \frac{U'''(w+Ex)}{3!}\mu_3 + \ldots\ldots$$

If other factors are held constant, then the higher the σ_χ^2, the lower the expected utility of a risk averter, because $U'' < 0$ and the higher the skewness, the higher the expected utility as long as $U''' > 0$. Therefore, if $U'' < 0$ the investor will dislike the variance and if $U''' > 0$ then the investor will dislike negative skewness and like positively skewness.

3.7.4.2 The Relationship between Positive Skewness and Decreasing Absolute Risk Aversion (DARA).

Another rationale for the assumption that $U''' > 0$ relies on the observation that the higher the investor's wealth the smaller the risk premium that he or she will be willing to pay to insure a given loss. As we have already described Arrow and Pratt determined that the risk premium is given by the following formula:

$$A(W) = -\frac{\sigma^2}{2}\frac{U''(w)}{U'(w)} \qquad (3.20)$$

It has been observed that the larger the wealth, the smaller the average amount of risk premium that the investor will be willing to give up in order to get rid of the risk. Formally, this claim is that $A'(W) < 0$. Using the above definition of $A(W)$, this means that the following must hold:

$$A'(W) = -\frac{\sigma^2}{2}\frac{U'(W)U'''(W) - [U''(W)]^2}{[U'(W)]^2} < 0 \qquad (3.21)$$

and this can hold only if $U'''(W) > 0$.

To sum up participation in a lottery and buying insurance provides some evidence that $U'''(W) > 0$. The empirical studies and the observation that $A'(W) < 0$ provide much stronger evidence for the preference for positive skewness and aversion to negative skewness, which in turn, strongly support the hypothesis that $U'''(W) > 0$. This evidence is strong enough to make it worthwhile to establish an investment decision rule for $U \in \boldsymbol{U_3}$ where $U' \geq 0, U'' \leq 0$ and $U''' \geq 0$.

3.7.4.3 The Third Stochastic Dominance Rule

The optimal investment rule for $U \in \boldsymbol{U_3}$ is given in the following Theorem.
$Theorem$ 3: Let F(x) and G(x) be the cumulative distributions of two investments whose density functions are $f(x)$ and $g(x)$, respectively. Then F dominates G by third degree stochastic dominance (TSD), which we denote by FD_3G if and only if the following two conditions hold:

$$\int_{-\infty}^{x}\int_{-\infty}^{u}[G(t) - F(t)]dtdu \geq 0 \ \ for\ all\ values\ of\ x, \qquad (3.22)$$

$$E_F(x) \geq E_G(x) \qquad (3.23)$$

and there is at least one x_0 for which a strong inequality holds. This Theorem can also be stated as follows:

$$\int_{-\infty}^{x}\int_{-\infty}^{u}[G(t)-F(t)]dtdu \geq 0 \ \ for\ all\ x, and\ E_F(x) \geq E_G(x) \Leftrightarrow$$

$$\Leftrightarrow \quad E_F U(x) \geq E_G U(x) \quad for\ all\ U \in \boldsymbol{U_3} \qquad (3.24)$$

We call such a dominance third-degree because assumptions of the third order are made on U (i.e, $U' > 0, U'' \leq 0$ and $U''' \geq 0$). It is worth to mention that a preference of one investment over another by TSD may be due to the preferred investment having a higher mean, a lower variance or a higher positive skewness.

3.7.4.4 Sufficient Rules and Necessary Rules for TSD

TSD is a necessary and sufficient decision rule for all $U \in \boldsymbol{U_3}$. However, we can establish various sufficient rules and necessary rules for $U \in \boldsymbol{U_3}$ dominance. We will consider here three of such sufficient rules.

- Sufficient Rule 1: The FSD rule is a sufficient rule for TSD. If FD_1G, then $F(x) \leq G(x)$ for all x with at least one strong inequality. This implies that: $E_F(x) > E_G(x)$ and $\int_{-\infty}^{x}\int_{-\infty}^{u}[G(t)-F(t)]dtdu \geq 0$ because FSD implies that the superior investment has a higher mean and that $G(x) - F(x)$ is non-negative.

- Sufficient Rule 2: SSD is a sufficient rule for TSD. If FD_2G then:

  $$\int_{-\infty}^{x}[G(t)-F(t)]dt \geq 0 \ \ for\ all\ values\ of\ x,$$

  Then:

  $$\int_{-\infty}^{x}\int_{-\infty}^{u}[G(t)-F(t)]dtdu \geq 0$$

  Thus, FD_2G implies that the two conditions required for TSD dominance hold; hence FD_3G.
  To add one more explanation for these sufficiency rules recall that:

$FD_1G \Longrightarrow E_F U(x) > E_G U(x)$ for all $U \in \boldsymbol{U_1}$ and because $\boldsymbol{U_1} \supseteq \boldsymbol{U_3}$, it is obvious that $E_F U(x) > E_G U(x)$ for all $U \in \boldsymbol{U_3}$. A similar explanation holds for the sufficiency of SSD because $\boldsymbol{U_2} \supseteq \boldsymbol{U_3}$.

Of course, many more sufficient rules are possible. However, the most important sufficient rules for $U \in \boldsymbol{U_3}$ are the FSD and SSD conditions. We now turn to the necessary rules which imply TSD.

- Necessary Rule 1 → *The means*. Unlike FSD and SSD, TSD explicitly requires that $E_F(x) \geq E_G(x)$ in order to have FD_3G. This condition on the expected values is a necessary condition for F over G in U_3. Note that for FSD and SSD we had to prove that this condition was necessary for dominance but for TSD, there is nothing to prove because it is explicitly required by the dominance condition.

- Necessary Rule 2 → *The Geometric Means*.
  Suppose that FD_3G.
  Then: $E_F(\log(x)) \geq E_G(\log(x))$ because $U_0(x) = \log(x) \in U_3$.
  Hence the geometric mean of F must be greater or equal to the geometric mean of G is a necessary condition for dominance in $\boldsymbol{U_3}$.

- Necessary Rule 3 → *The ''Left Tail'' condition.* Like FSD and SSD, for FD_3G, the left tail of the cumulative distribution of G must be "thicker" than the left tail of F. In other words, $Min_F(x) \geq Min_G(x)$ is a necessary condition for FD_3G.

3.7.5 Stochastic Dominance with a Riskless Asset

The difficulty with Stochastic Dominance rules is that, in general, it results in a large efficient set of investments, i.e in many cases this framework is unable to rank the two risky investments under consideration. The well-known mean-variance rule suffers from a similar drawback. However, when riskless borrowing or lending is allowed, the mean variance approach provides a sharper decision which makes it possible to derive an equilibrium pricing relationship known as the capital assets prices model (CAPM). Similar ideas are also applied to Stochastic Dominance.
When the riskless asset is allowed, the Stochastic Dominance analysis is denoted by SDR. We say that F dominates G in the SDR framework, if and only if, for every

element in $\{G_\beta\}$ there is at least one element in $\{F_\alpha\}$ which dominates it in the Stochastic Dominance framework, where $\{F_\alpha\}$ and $\{G_\beta\}$ contain all the linear combinations of the riskless and risky assets given by the $\{F_\alpha\} = aX + (1 - a)r$ and $\{G_\beta\} = \beta X + (1 - \beta)r$, respectively. We denote the corresponding FSD, SSD and TSD rules with a riskless asset by $FSDR, SSDR$ and $TSDR$, respectively.

SDR rules provide a sharper decision relative to Stochastic Dominance rules, hence a smaller efficient set of risky investments is obtained. To see the intuition of this results let us focus on the comparison of FSD and $FSDR$. Suppose that two cumulative distributions F and G intersect, hence no FSD prevails. By mixing F with the riskless asset one creates a new distribution F_α. F_α rotates about the vertical line $X = r$. If one can find a mix such that F_α is located completely below G, we have FSD of F_α over G. Levy and Kroll (1978) show that if such an α exists, then, for any selected β, one can find $\gamma = \alpha\beta$ such that F_γ dominates G_β and hence F dominates G by $FSDR$. Thus F may dominate G with a riskless asset even of dominance does not exist in the absence of the riskless asset. This implies that the SDR efficient sets are no larger than the Stochastic Dominance efficient sets. Similar intuitive explanations hold in the comparisons of SSD and $SSDR$, and TSD and $TSDR$.

To summarize this section, we must note the following two results: It is well known that $FD_iG\ (i = 1,2,3)$only if $E_F \geq E_G$. Such a requirement is not necessary for dominance by $FSDR, SSDR$ or $TSDR$, since the mean of the risky portfolio can be changed by altering the proportion of the riskless asset in the mixed portfolio. Similarly, for Stochastic Dominance, a necessary condition for F to dominate G is that its left tail must be located to the right of the left tail of the distribution of G. Such a necessary condition is not required for SDR since by mixing F with the riskless asset, F rotates about the vertical line $x = r$ so that the left tail of the new distribution is below G.

The Stochastic Dominance and SDR criteria are related as follows:

$$
\begin{tikzcd}
FSD \arrow[r, Rightarrow] \arrow[d] & SSD \arrow[r, Rightarrow] \arrow[d] & TSD \arrow[d] \\
FSDR \arrow[r, Rightarrow] & SSDR \arrow[r, Rightarrow] & TSDR
\end{tikzcd}
$$

Since all these rules are transitive, it is obvious that FSD implies $TSDR$ and $SSDR$ and SSD implies $TSDR$. Thus, the $TSDR$-efficient set must be a subset of all the other efficient sets derived by either the $SD\ or\ SDR$ criteria.

Chapter 4

Empirical Analysis

4.1 Empirical Analysis

In this study, we use the concepts of first-order stochastic dominance, second-order stochastic dominance and third-order stochastic dominance as well as the $CVaR$ approach, including three different investment tactics, in order to construct optimal portfolios. In particular, we propose to determine the optimal portfolios based on the FSD, SSD and FSD criteria to find the optimal portfolio weights. We implement all the alternative models in the General Algebraic Modeling System (GAMS). In constructing our *FSD-based* portfolio we adopt 0-1 Mixed Integer Linear Programming developed in Kuosmanen (2004), as well as the construction of our *SSD-based* and *TSD-based* portfolios are formulated in terms of standard Linear Programming developed once again in Kuosmanen (2001,2004). Furthermore, in order to compare the performance of the optimal competing portfolios, we evaluate all these alternative portfolios with respect to the market benchmark portfolio using several performance measures such as the Sharpe Ratio, the Sortino Ratio, the opportunity cost and portfolio turnover.

In the empirical tests we consider investments in the US market. We want to construct several optimal portfolios based on alternative strategies. We use data on monthly closing prices of S&P500, as well as a number of stocks obtained by Datastream. We choose assets from different sectors as these have been categorized by The Global Industry Classification Standard (GICS) and thus a total number of 30 assets are concerned in each portfolio. With respect to the sectors of the alternative asset classes we have chosen the following large-cap companies stocks (based on their market capitalization): Microsoft Corp., Oracle Corp. and Intel Corp. from the Information Technology sector; Amazon Inc., McDonald's Corp., Wal-Mart stores derived from the Consumer Discretionary sector; Exxon Mobil, Chevron Corp., ConocoPhillips from the Energy sector; Medtronic plc, Pfizer Inc., Johnson&Johnson from the HealthCare sector; General Electric, MMM, Northrop Grumman Corp. from the Industrial sector; Procter & Gamble, PepsiCo Inc., Coca Cola Company derived from the Consumer Staples sector; AT&T Inc., Verizon Communications, CenturyLink Inc. from the Telecommunications Services sector; JPMorgan Chase & Co., Bank of America Corp., Wells Fargo from the Financial sector; American Electric Power, Southern Co., Sempra Energy derived from the Utilities sector and Ecolab Inc., Air Products & Chemicals Inc, PPG Industries from the Materials sector.

From the data of index prices we computed their corresponding monthly returns using the following formula:

$$r_t = \frac{PI_t - PI_{t-1}}{PI_{t-1}} \quad (4.1)$$

All time series have a monthly time-step and cover the period from December 1999 to July 2016, thus we have obtained 200 scenarios.

Table 4.1 reports summary descriptive statistics regarding the performance of the selected assets as well as the Benchmark over the simulation period. We can observe that Amazon exhibits the highest monthly average return however it also has the greatest standard deviation. General Electric appears to have the smallest value of mean return during the planning horizon. PepsiCo has the lowest standard deviation among all the assets. Moreover, in all cases there is a positive correlation among the selected assets with the Benchmark. In particular General Electric yields the highest correlation with the Market Index, while Southern seems to have the worst correlation, respectively.

Table 4.1: Descriptive Statistics

Entries report the descriptive statistics for the alternative asset classes as well as the Market Index(S&P500) used in this study. The mean returns, the standard deviations, the skewness, the kurtosis and the correlation between the selected assets and the Benchmark are reported. The dataset spans the period from December 1999 to July 2016.

	Average Return	Standard Deviation	Skewness	Kurtosis	Correlation
Market Index	0,451	4,328	-0,519	3,944	1
Microsoft	0,579	8,919	0,347	5,956	0,592
Oracle	0,731	9,985	0,219	6,334	0,533
Intel	0,657	10,241	-0,515	5,284	0,620
Amazon	**2,180**	**14,407**	0,437	5,767	0,497
McDonalds	0,908	5,836	-0,461	5,389	0,494
WalMart	0,320	5,554	-0,214	4,219	0,304
ExxonMobil	0,716	4,973	0,339	4,578	0,467
Chevron	0,892	6,029	0,174	3,824	0,565
ConocoPhillips	0,989	7,773	-0,103	3,899	0,559
Medtronic	0,748	6,155	-0,130	5,484	0,472
Pfizer	0,472	5,767	-0,100	2,959	0,483
Johnson&Johnson	0,796	4,741	-0,235	4,567	0,388
GeneralElectric	**0,293**	7,637	-0,174	4,616	**0,730**
MMM	1,019	5,742	0,098	3,883	0,576
NorthropGrumman	1,508	6,782	0,132	6,494	0,376
Procter&Gamble	0,603	5,271	-1,778	13,405	0,213
Pepsico	0,854	**4,525**	-0,358	5,830	0,424
Cocacola	0,543	5,118	-0,379	4,043	0,383
AT&T	0,535	6,663	0,169	4,949	0,426
Verizon	0,570	6,849	0,934	8,106	0,455
Centurylink	0,410	8,118	-0,039	8,036	0,445
JPMorgan	0,805	9,097	-0,187	3,987	0,679
BankofAmerica	0,748	12,005	0,454	11,702	0,545
Wellsfargo	1,011	7,984	0,111	10,270	0,476
AmerElecPwr	0,930	5,978	-0,165	4,108	0,308
Southern	1,151	4,705	0,309	6,213	**0,064**
Sempra	1,325	5,551	-0,222	3,994	0,331
Ecolab	1,192	5,833	0,130	6,949	0,499
AirPrds&Chems	1,184	6,670	-0,125	4,394	0,643
PPG Industries	1,082	6,989	0,212	3,929	0,706

4.2 *Constructing optimal portfolios using $CVaR$ optimization model*

We examine the performance of various decision strategies in static as well as in dynamic tests to identify the most promising tactic. In static test we constructed the risk-return profile *(efficient frontier)* generated with appropriate variants of the model at a certain point in time. The static test considered $CVaR$-related portfolio selection problems. The models were repeatedly run for various levels of minimum expected return.

The solutions trace the corresponding efficient frontier of expected portfolio return vs. the $CVaR$ risk metric of portfolio losses (at $\alpha = 95\%$) over the planning horizon. The efficient frontier, depicted in *Figure 4.1*, is determined in-sample; that is, with respect to the postulated scenarios.

In order to find the lowest point of the portfolios efficient frontier we maximized $CVaR$ without any constraint. Next we maximized expected return again without any constraint in order to get the upper point and finally we have to include more points to get a detailed view of the portfolios efficient frontier. To accomplish that we calculate the width from the formula $E(R)_{max} - E(R)_{min}$ and then we divide it by 10 to get different weighted values of the expected target return. The results are given in *Table 4.2.*

Table 4.2: Entries report the selected values of the Expected Return and the $CVaR$ in order to construct the Efficient Frontier.

CVaR	E(R)
-0.05713	0.01001
-0.06030	0.01119
-0.06472	0.01237
-0.07005	0.01355
-0.08066	0.01473
-0.09768	0.01591
-0.12170	0.01708
-0.15823	0.01826
-0.20136	0.01944
-0.25224	0.02062
-0.41200	0.02180

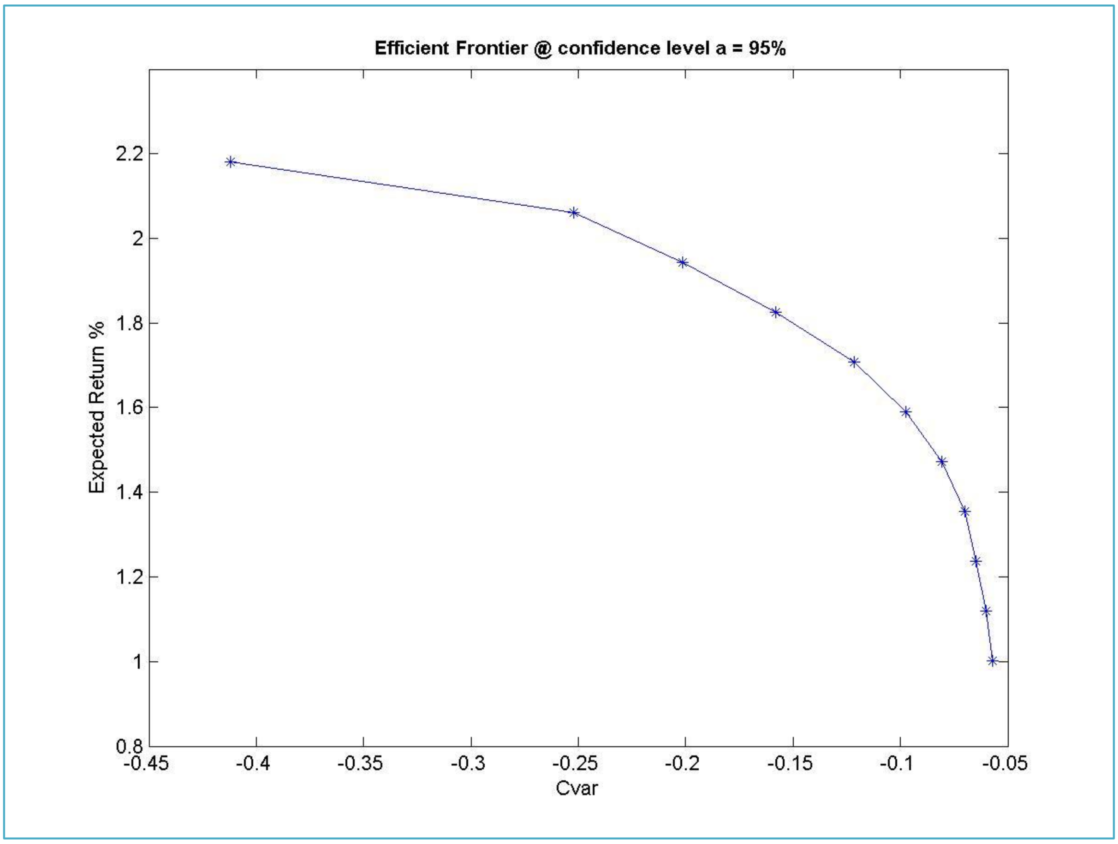

Figure 4.1: Risk-return frontier of portfolios generated with the CVaR model.

Our next goal was to conduct dynamic tests (backtesting experiments over the last 120 months) for three different investment strategies. The rolling horizon simulations cover the 120-month period from 08/2006 to 07/2016. At each month, we use the historical data from the previous 6 years (~80 monthly observations).We then solve the resulting optimization model and record the optimal portfolio. The clock is advanced one month and the realized return of the portfolio is determined from the actual market values of the assets. The same procedure is then repeated for the next time period and the ex post realized returns are computed. We ran such backtesting experiments for the $CVaR$ model using various values of target return.
First we maximize the $CVaR$ without any constraint for the defensive investor, due to the fact that this strategy corresponds to the minimum risk case. The next model we examined was the aggressive investor in which we maximized the expected return again without any constraint; in this strategy we want to gain the maximum benefit without any consideration on risk value. The last strategy examined in this study was the average investor model; to succeed this we maximized $CVaR$ subject to the expected target return. In every dynamic test we have proceed so far, we have found a different target return constraint, since it is calculated through the formula:

$$\mu = \frac{E(R)_{max} + E(R)_{min}}{2} \qquad (4.2)$$

The values of $E(R)_{max}$ *and* $E(R)_{min}$ are different in every optimization problem. In this model our purpose was to have a good ratio between expected return and risk value. The results of all the $CVaR$-related strategies are depicted in *Figure 4.2.*

4.3 Constructing optimal portfolios using FSD, SSD and TSD criteria

4.3.1 Preliminaries

In general, we can think of the Stochastic Dominance concepts as properties of the probability distributions. Speaking of diversification, however, it contributes to our understanding to assume that the rate of return of an investment portfolio as our random variable. Consider two risky portfolios j and k with the return distributed according to the cumulative distribution functions $G_j \ and \ G_k, G_j \neq G_k$, respectively.
$Definition$ 1: Portfolio j dominates portfolio k by FSD, SSD and TSD, denoted by jD_1k, jD_2k, jD_3k respectively, if and only if:

$$FSD\colon\ G_k(x) - G_j(x) \geq 0 \ \ \forall\, x \in R \ \ and \ \ G_k(x) - G_j(x) > 0 \quad for\ some\ x \in R \qquad (4.3)$$

$$SSD\colon\ \int_{-\infty}^{x} [G_k(t) - G_j(t)]dt \geq 0 \ \ \forall\, x \in R \quad and \quad \int_{-\infty}^{x} [G_k(t) - G_j(t)]dt > 0 \quad for\ some\ x \in R \quad (4.4.)$$

$$TSD\colon\ \int_{-\infty}^{x} \int_{-\infty}^{u} [G_k(t) - G_j(t)]dtdu \geq 0 \ \ \forall\, x \in R \ \ and \quad \int_{-\infty}^{x} \int_{-\infty}^{u} [G_k(t) - G_j(t)]dtdu > 0 \ \ for\ some\ x \in R \ (4.5)$$

The Stochastic Dominance criteria have the following well-known economic interpretation as we have already discussed in the previous chapter, in terms of the Expected Utility Theory: Consider a continuously differentiable Bernoullian utility function $U\colon R \to R$. If the investor is non-satiated then jD_1k, implies that the investor prefers portfolio j over k. If the investor is risk-aversive in addition to non-satiation, then jD_2k, implies that preference of portfolio j over k and conversely. Furthermore if the investor exhibits decreasing absolute risk-aversion then, jD_3k, implies that portfolio j is preferred over k. Converse relationships also hold: If the investor prefers portfolio j over k whenever jD_1k, then the investor is non-satiated. If the investor prefers portfolio j over k whenever jD_2k, then the investor is risk aversive. Finally, if the investor prefers portfolio j over k whenever jD_3k, then the investor exhibits decreasing absolute risk aversion.

In an empirical portfolio analysis, the underlying probability distributions must be estimated from the available data and to this end it is natural to consider a finite and therefore discrete sample of return observations of the N assets from T time periods indexed as $i \equiv \{1,2, \dots \dots, N\}$ and $t \equiv \{1,2, \dots \dots ., T\}$, respectively. This gives panel data represented by matrix $Y \equiv (y_1 \dots\dots y_N)$ with $y_j \equiv (y_{j1} \dots \dots y_{jT})^{\mathrm{T}}$. Portfolios can be modeled in terms of portfolio weights. These portfolio weights are denoted by a column vector $\lambda \in \Lambda$, where $\Lambda \subset \{\lambda \in \mathbb{R}^N \,|\sum_{i=1}^{N} \lambda_\iota = 1\}$ represents their feasible domain, which is assumed to be closed and bounded. The market set spanned by Λ is $\Psi \equiv \{y \in \mathbb{R}^T | y = Y\lambda; \lambda \in \Lambda\}$. In order to derive the empirical distribution function for an arbitrary portfolio i, we can sort each column vector y_i in ascending order, and denote the resulting ranked return vector by $x_i\colon x_{i1} \leq x_{i2} \leq \cdots \leq x_{iT}$. Using ranked vectors x, the EDF for asset j is a step function characterized as $H_n(r) \equiv \max\{t \in \tau | r \geq x_{nt}\}/T$. The EDF is a non-parametric, minimum-variance unbiased estimator of the underlying unobservable CDF. Moreover, the standard approach is to examine Stochastic Dominance efficiency in terms of H_n. In order to distinguish the application of the Stochastic Dominance criteria to $EDFs$ from the theoretical Stochastic Dominance conditions of Definition 1, symbol $\widehat{D_l} = l = 1,2,3$ is used for Stochastic Dominance relations when an EDF is used to estimate the unknown CDF.

$Theorem$ 1: The following equivalence results hold for empirical distribution functions of all portfolios j and k:

$$FSD\colon\quad j\widehat{D_1}k \;\Leftrightarrow x_{jt} \geq x_{kt} \;\forall\, t \in T,\;\; and$$

$$x_{jt} > x_{kt} \;\; for\, some\; t \in T \qquad (4.6)$$

$$SSD\colon\quad j\widehat{D_2}k \;\Leftrightarrow\; \sum_{i=1}^{t} x_{ji} \geq \sum_{i=1}^{t} x_{ki} \;\;\forall\, t \in T, and$$

$$\sum_{i=1}^{t} x_{ji} > \sum_{i=1}^{t} x_{ki}\, for\, some\; t \in T \qquad (4.7)$$

$$TSD\colon\quad j\widehat{D_3}k \Leftrightarrow \sum_{i=1}^{t}\sum_{\nu=1}^{i} x_{ji} \geq \sum_{i=1}^{t}\sum_{\nu=1}^{i} x_{ki} \;\;\forall\, t \in T\; and$$

$$\sum_{i=1}^{t}\sum_{\nu=1}^{i} x_{ji} > \sum_{i=1}^{t}\sum_{\nu=1}^{i} x_{ki} \;\; for\, some\; t \in T \qquad (4.8)$$

Focusing on the market set Ψ, Stochastic Dominance efficiency is characterized by the following definition:

$Definition\ 2$: Portfolio $k \in \Psi$ is $FSD(SSD, TSD)$ in market set Ψ if and only if $j\widehat{D_1}k\big(j\widehat{D_2}k, j\widehat{D_3}k\big) \Rightarrow j \notin \Psi$; otherwise k is $FSD(SSD, TSD)$ inefficient.
It is standard knowledge that the Stochastic Dominance efficiency criteria are transitive in the following sense: TSD efficiency implies SSD efficiency, which in turn implies FSD efficiency. Conversely stated, FSD inefficiency implies SSD inefficiency, which in turn implies TSD inefficiency. However, FSD, SSD and TSD criteria are generally not equivalent. The FSD efficiency criterion is generally the weakest one in terms of discriminatory power, involving the largest efficient subset of Ψ. Proceeding towards the higher order efficiency criteria can generally improve the discriminatory power of the Stochastic Dominance test since the Stochastic Dominance efficient subsets become smaller. However, the greater power of the efficiency tests should be balanced against the additional restrictions concerning the risk-preferences of the investor, as we will discuss above.

4.3.2 Methodology

Consider a specified benchmark-portfolio (Benchmark) which is held for a finite time period. For the same time period, we construct alternative FSD-based, SSD-based and TSD-based portfolios using the Kuosmanen (2001, 2004) linear programming approach as well as the Kuosmanen 0-1 mixed integer linear programming approach. In Kuosmanen, the procedure is based on the necessary condition for Stochastic Dominance efficiency; and it measures the degree of inefficiency for the benchmark portfolio in terms of FSD, SSD and TSD respectively. To this end, we employ a "rolling-sample" approach. Assume that the dataset consists of T (in our case $T = 200$) monthly observations for each asset and K is the size of the employed rolling window used for the calculation of the portfolio weights. Standing at each month t, we use the previous K observations to estimate the asset allocation weights that maximize the expected utility. Next, the estimated weights are used to compute the out-of-sample realized return over the period $[t, t + 1]$. We repeat this procedure by incorporating the return of the next period and ignoring the earliest one, until the end of the sample period. Then, we use the time series of realized portfolio returns to evaluate the out-of-sample performance of the derived alternative portfolios. Once again, we choose the size of the rolling window $K = 80$. This delivers December 1999- July 2006 as the starting interval for the estimation of optimal portfolio weights and August 2006-July 2016 as the out-of-sample period. The performance of the portfolios is compared with the benchmark's out-of sample return based on traditional performance measures, all of which we will discuss below. We impose short-sale constraints in the portfolio selection process, which ensures that all our constructed portfolios are feasible choices for delegated money management structures, where shorting is frequently not allowed. Thus, portfolio

weights are restricted to be nonnegative and sum to one for each of the considered portfolios.
In constructing our FSD-based, SSD-based and TSD-based portfolios, we have already mentioned that we adopt the algorithms developed in Kuosmanen. This approach allows for testing if a benchmark portfolio return distribution is FSD, SSD and TSD efficient respectively, relative to a given asset span. If the benchmark is not efficient, the solution also delivers a vector of portfolio weights corresponding to a well-diversified FSD-efficient, SSD-efficient and TSD-efficient portfolio that first-order, second-order, as well as third-order, stochastically dominates the benchmark.
Below we formally present the 0-1 mixed integer linear programming formulations of Kuosmanen, as well as the linear programming formulations, with notation changed to conform to our study and constraints specified to match our requirements of nonnegative weights. In all specifications, the asset span consists of N assets with T monthly return observations y_{it} each. The vector of portfolio weights to be optimized is denoted by λ.
We implement the following 0-1 mixed integer linear programming procedure of Kuosmanen (2004) in order to construct the FSD-efficient portfolio:

$$\theta_1(y_{bench}) = max_{\lambda,P}\left(\sum_{t=1}^{T}\sum_{i=1}^{N} y_{i,t}\,\lambda_i - \sum_{t=1}^{T} y_{bench,t}\right)\frac{1}{T}$$

$$subject\ to\ \sum_{i=1}^{N} y_{i,t}\,\lambda_i \geq \sum_{j=1}^{T} P_{t,j} y_{bench,t}\ \ for\ t = 1,2, \ldots \ldots, T$$

$$P \in \left\{[P_{i,j}]_{T\times T} | P_{i,j} \in \{0,1\}; \sum_{i=1}^{T} P_{i,j} = \sum_{j=1}^{T} P_{i,j} = 1\ \ \forall\ \ i,j = 1, \ldots, T\right\} \quad (4.9)$$

$$0 \leq \lambda_i \leq 1, \forall i = 1, \ldots, N$$

$$\sum_{i=1}^{N} \lambda_i = 1$$

where we use $y_{bench,t}$ to denote the benchmark portfolio return at time t, and P is a permutation matrix. The elements of this matrix are consisted of binary integers {0,1} and its rows and columns sum up to unity. Technically, permutation matrices allow us to sort the elements of a return vector in any arbitrary order. The vector of optimal portfolio weights λ from the above approach is used to construct the FSD-based portfolio.

Test statistic θ_1 has an intuitive interpretation as the measure of inefficiency of the evaluated portfolio that is the Benchmark. Statistic $\theta_1(y_{bench})$ indicates the maximum increase in mean return obtainable without aggravating the risk exposure of the portfolio. The main difference between the SD-based measure θ_1 and the traditional Mean-Variance measures lies in the treatment of risk. While the Mean-Variance approach can employ variance as a measure for risk, no such quantitative measure exists for the Stochastic Dominance. Rather, the Stochastic Dominance approach must rely on its partial preference orderings, which offer a qualitative criterion for risk. Moreover, a positive value of the SD statistic θ_1 implies that there exists a portfolio that dominates the evaluated portfolio (i.e. the Benchmark) by FSD.

The FSD dominance implies that every non-satiated investor, irrespective of his risk preferences, would be better off by holding λ that yields $\theta_1(y_{bench})$ units higher mean return. In this sense, $\theta_1(y_{bench})$ accounts for risk without explicitly quantifying it. Statistic $\theta_1(y_{bench})$ can be interpreted as the maximum loss of mean return due to FSD inefficiency.

We now turn to the implementation of the linear programming, concerning the construction of the SSD-efficient portfolio. We proceed to the following algorithm of Kuosmanen (2004):

$$\theta_2(y_{bench}) = max_{\lambda,W}\left(\sum_{t=1}^{T}\sum_{i=1}^{N} y_{i,t}\,\lambda_i - \sum_{t=1}^{T} y_{bench,t}\right)\frac{1}{T}$$

$$subject\ to\ \sum_{i=1}^{N} y_{i,t}\,\lambda_i \geq \sum_{j=1}^{T} W_{t,j} y_{bench,t}\ \ for\ t = 1,2, \dots \dots, T$$

$$W \in \left\{[W_{i,j}]_{T\times T} | 0 \leq W_{i,j} \leq 1; \sum_{i=1}^{T} W_{i,j} = \sum_{j=1}^{T} W_{i,j} = 1\ \ \forall\ \ i,j = 1, \dots, T\right\} \qquad (4.10)$$

$$0 \leq \lambda_i \leq 1, \forall\ i = 1, \dots, N$$

$$\sum_{i=1}^{N} \lambda_i = 1$$

Once again, we use $y_{bench,t}$ to denote the benchmark portfolio return at time t, and W is a doubly stochastic matrix. In the previous algorithm, in order to obtain a necessary test of SSD efficiency, we simply relax the binary integer constraint from the permutation matrix P, and hence take a liberty to form convex combinations of the elements of vector $y_{bench,t}$. The elements of this doubly stochastic matrix must be nonnegative real numbers and its rows and columns sum up to unity. The logic of this test is analogous to that of the FSD test. The test statistic $\theta_2(y_{bench})$ can be interpreted as the inefficiency measure in the mean sense, similar to the FSD case. This measure indicates the maximum increase of mean return that could be obtained by choosing an efficient portfolio from the subset that dominates the evaluated portfolio (i.e. the Benchmark) by SSD. The above LP algorithm, generally, tests only for the necessary condition of SSD.

Finally, in order to construct the TSD-efficient portfolio, we implement the following LP procedure of Kuosmanen (2001). At this point, we must first introduce the subsequent auxiliary vectors:

$$z^{kl} \equiv \left(z_1^{kl} \ldots\ldots z_T^{kl}\right); k < l; k, l \in T$$

$$where \qquad z_i^{kl} \equiv \begin{cases} y_{bench,i}, i = 1, \ldots, k-1 \\ r_{kl}, i = k, \ldots, l \\ \rho_{kl}, i = l+1 \\ y_{bench,i}, i = l+2, \ldots, T \end{cases} ; \qquad (4.11)$$

$$r_{kl} \equiv \frac{\sum_{i=k}^{l}(l-i+1)y_{bench,i}}{\sum_{i=k}^{l}(l-i+1)} \; and$$

$$\rho_{kl} \equiv \frac{\sum_{i=1}^{l+1}(l-i+2)y_{bench,i}}{\sum_{i=1}^{k-1}(l-i+2)y_{bench,i} + (k-l)r_{kl}}$$

These auxiliary vectors have identical elements to y_{bench} , i.e. the return vector of the evaluated portfolio, except for the elements from k through $l+1$. For the elements from k to l these vectors contain a constant (risk-free) value of r_{kl}, and the value of element $l+1$ is ρ_{kl}. Note that in the special case of $k = 1, l = T$ we have:

$$z^{1T} \equiv (z_{1T} \ldots\ldots z_{1T});$$

$$and \;\; r_{1T} = \frac{\sum_{i=1}^{T}(T-i+1)y_{bench,i}}{\sum_{i=1}^{T}(T-i+1)}, \qquad (4.12)$$

which turns out the smallest risk-free return that dominates the evaluated portfolio by TSD.

In addition, analogous to matrix W, we define the following additional $T \times T$ weight matrices $W_{i,j}^{0}, W_{i,j}, i < j; i,j \in T$.

Concerning the construction of the TSD-efficient portfolio, we perform the below linear programming procedure of Kuosmanen (2001):

$$\theta_3(y_{bench}) = max_{\lambda,W,W^0}(\sum_{t=1}^{T}\sum_{i=1}^{N} y_{i,t}\,\lambda_i - \sum_{t=1}^{T} y_{bench,t})\frac{1}{T}$$

$$subject\ to\ \sum_{i=1}^{N} y_{i,t}\,\lambda_i \geq \sum_{j=1}^{T} W_{t,j}^{0} y_{bench,t} + \sum_{j=1}^{T} W_{t,j} z_t\ \ for\ t = 1,2,\dots\dots,T$$

$$W, W^0 \in \left\{ \begin{array}{c} [W_{i,j}]_{T\times T}, [W_{i,j}^{0}]_{T\times T} \middle| 0 \leq W_{i,j} \leq 1, 0 \leq W_{i,j}^{0} \leq 1; \\ \sum_{i=1}^{T}(W_{i,j} + W_{i,j}^{0}) = \sum_{j=1}^{T}(W_{i,j} + W_{i,j}^{0}) = 1\ \ \forall\ \ i,j = 1,\dots,T \end{array} \right\} \quad (4.13)$$

$$0 \leq \lambda_i \leq 1, \forall\ i = 1,\dots,N$$

$$\sum_{i=1}^{N} \lambda_i = 1$$

where we use $y_{bench,t}$ to denote the benchmark portfolio return at time t, and z_t the auxiliary vector which contains identical elements to $y_{bench,t}$ with some additional constrains. As long as we want to construct a TSD-efficient portfolio, which stochastically dominates the benchmark, we impose a stronger limitation considering the two matrices. In particular, we want the elements of these two matrices to be nonnegative real numbers and their rows and columns must sum up to unity; hence to create a new doubly stochastic matrix.

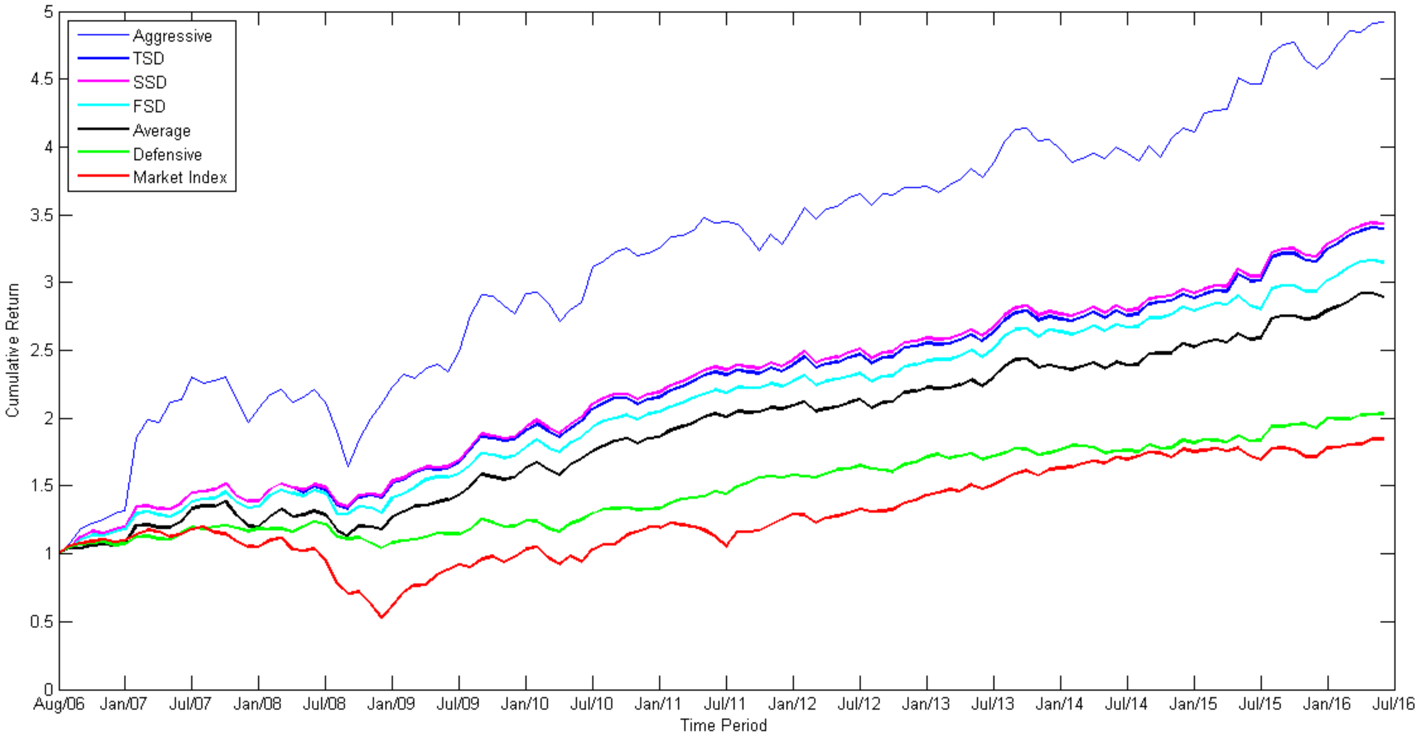

Figure 4.2: Ex-post realized performance of all the alternative competing portfolios. Backtesting simulations over the period August 2006-July 2016

The results of all the alternative competing portfolios are depicted in *Figure 4.2*. As we can observe in *Figure 4.2*, the minimum risk portfolio (i.e. Defensive-portfolio) attains a stable growth path over the planning horizon without any significant levels of volatility. In particular, this model simply optimizes the $CVaR$ risk measure, without imposing any constraint on the expected portfolio return. This is because we want the risk levels to be as low as possible. In other words we play it safe. This optimal portfolio is affected mush less than all the alternative portfolios during market downturns (e.g. July 2008-July 2009, March 2011-November 2011). Moreover, the Average-portfolio follows the same increasing growth path as the Defensive, whereas differs in his gains. Considering this particular investment tactic, we can observe that it achieves quite stable growth paths, with slight losses in only few instances during the simulation period. In this case, the Average portfolio yields a clearly superior performance compared to the Defensive-portfolio. The main reason for this outcome is that we want to have a good ratio between expected return and risk exposure. On the other hand regarding the Aggressive portfolio shows a noticeable improvement in performance; although this specific investment strategy exhibits the highest fluctuations in returns, reflecting a riskier portfolio, compared to the previous two tactics, it results in higher cumulative returns. Note that during the period July 2008-September 2008 this strategy yields substantial losses relative to the previous two strategies that attain a quite stable growth path over the same time period. Therefore, we obtain greater levels of volatility in comparison with its counterparts. The explanation for this behavior is that we want to maximize the expected return at any given level of risk exposure. In general, we can clearly state that all these alternative investment tactics concerning the $CVaR$ optimization model outperform the Benchmark, during the whole planning simulation horizon.

Turning to the Kuosmanen-related strategies, we can observe that the FSD-Kuosmanen optimal portfolio exhibits stable portfolio returns throughout the simulation period, with small losses in only very few periods. This investment tactic results in improved performance compared to the Average-portfolio as well as the Defensive-portfolio, as long as it yields greater cumulative returns during the simulation period. Moreover, its realized return paths are discernibly more stable that the corresponding path of the optimal Aggressive-portfolio. Furthermore FSD-Kuosmanen portfolio lagged a bit behind, particularly in periods of market downturns (e.g. July 2008-January 2009, February 2010-March 2010, June2015-September2015). Finally, regarding the SSD-Kuosmanen optimal portfolio as well as the TSD-Kuosmanen optimal we can observe that these two strategies demonstrate very similar ex post performance; with the SSD-Kuosmanen portfolio being a very slight favorite. While these two competing portfolios perform similarly for the first year of the simulation (e.g. August 2006-August 2007), after that the SSD-Kuosmanen portfolio outperform the TSD-Kuosmanen optimal portfolio. Both these two strategies outperform the FSD-Kuosmanen optimal portfolio, due to the fact that they yield higher cumulative returns throughout the planning horizon. Note that SSD-Kuosmanen and TSD-Kuosmanen optimal portfolios achieve quite stable growth paths, with slight losses in only few instances during the simulations. We can clearly state that these two investment tactics exhibit superior performance compared not only to the Average-portfolio and to the Defensive-portfolio, but also compared to the Benchmark.

Figures 4.3-4.8 illustrate the composition of the alternative competing optimal portfolios throughout the simulation period. Observe that in the Aggressive case the expected return of the portfolio is maximized without any consideration on risk. Hence, the entire budget is allocated to the asset with the highest expected return. Therefore, the Aggressive-portfolio consists of assets from Sempra, ConocoPhillips and Amazon with a proportion of 100% invested in each asset during the entire planning horizon. For the Defensive-portfolio *Figure 4.4* shows that this investment strategy resorts to a more diversified portfolio that consists of a substantial number of assets, that is 25 of the 30 available assets. In particular the top ten assets that a defensive investor chooses to hold into his portfolio are the following: Intel, Amazon, McDonalds, Wal-Mart, ExxonMobil, Medtronic, P&G, Southern, Sempra and Ecolab, where McDonalds as well as Wal-Mart provide the highest proportion of wealth to invest in.

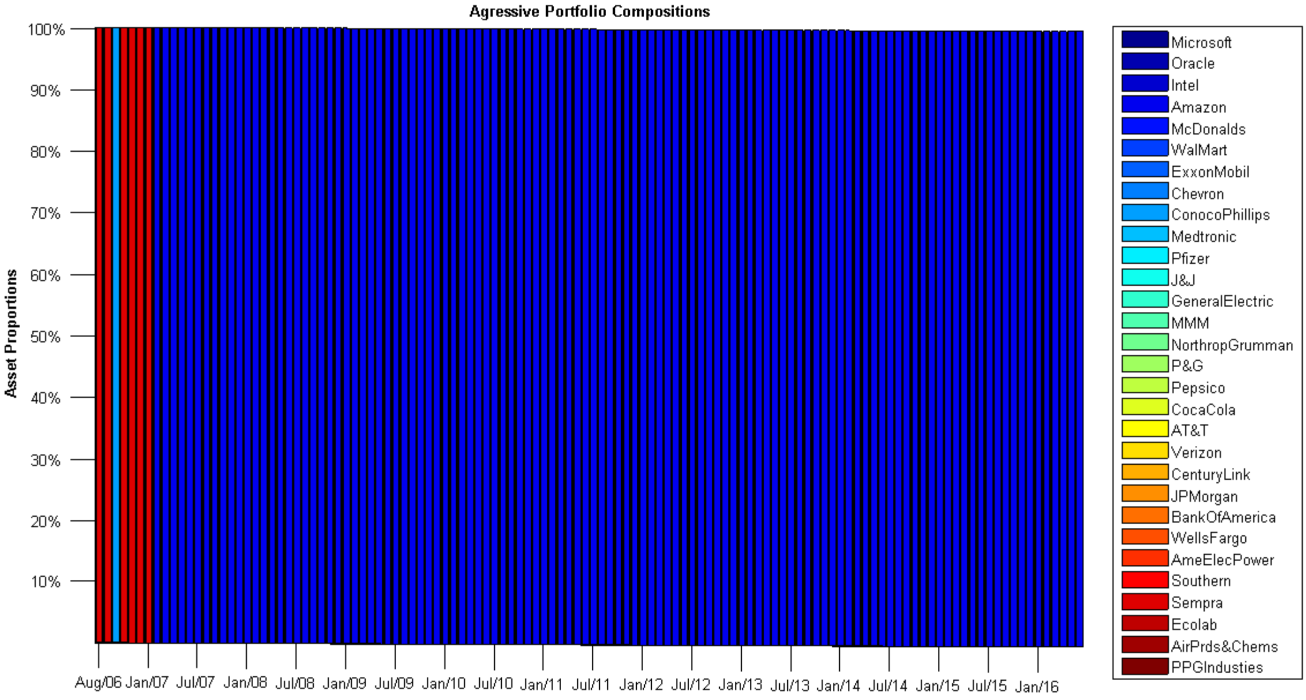

Figure 4.3: Composition of the Aggressive-portfolio during the backtesting period.

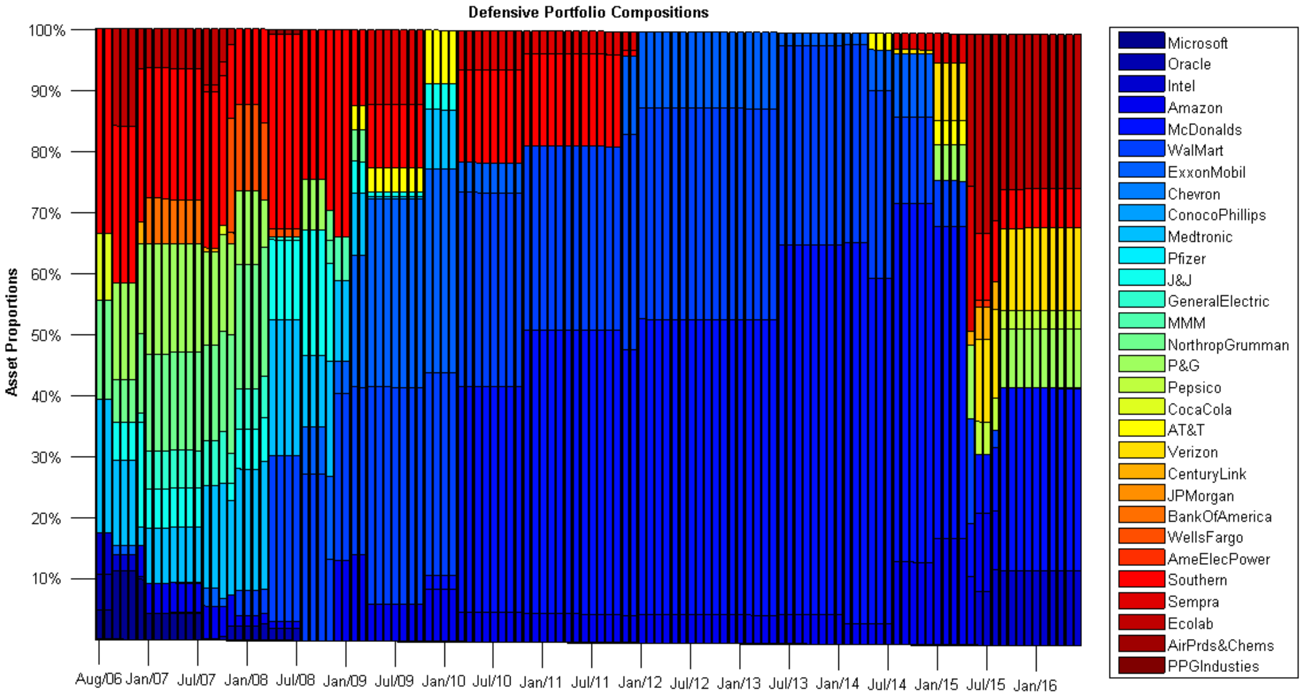

Figure 4.4: Composition of the Defensive-portfolio during the backtesting period.

Considering the Average-portfolio *Figure 4.5* shows that again this particular investment tactic propose a well-diversified portfolio that consists of an important number of assets; 25 of the 30 available assets. In particular the top ten assets that an average investor chooses to hold into his portfolio are the following: Amazon, McDonalds, Wal-Mart, ExxonMobil, ConocoPhillips, Northrop-Grumman, Verizon, Southern, Sempra and PPG Industries, where Amazon and McDonalds provide the highest proportion of wealth to invest in.

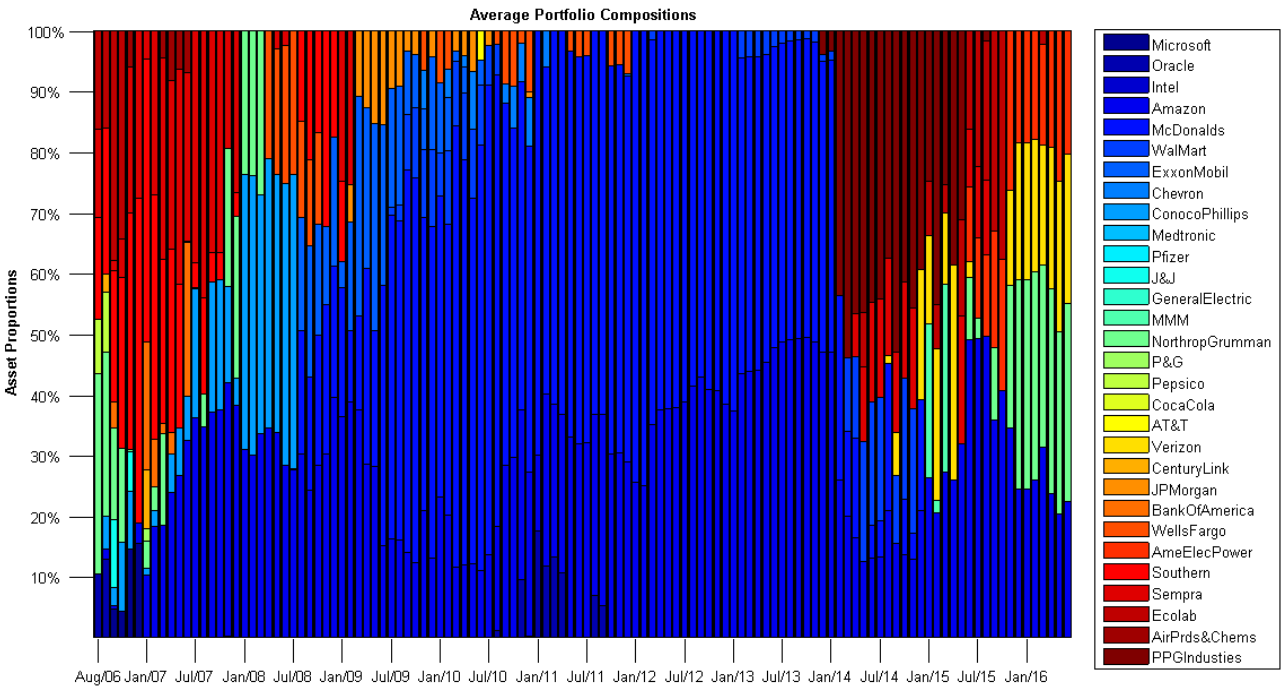

Figure 4.5: Composition of the Average-portfolio during the backtesting period.

Turning to the Kuosmanen-related strategies, we can observe from *Figure 4.6* that the FSD-Kuosmanen optimal portfolio yields a well-diversified portfolio, which is comprised of the entire set of the 30 assets. Particularly, the top ten assets that an investor with this portofolio holds are the subsequent: Amazon, McDonalds, Chevron, ConocoPhillips, Northrop-Grumman, American-Electric-Power, Southern, Sempra, Ecolab and PPG Industries, where Amazon and McDonalds exhibit the highest proportion of wealth to invest in.

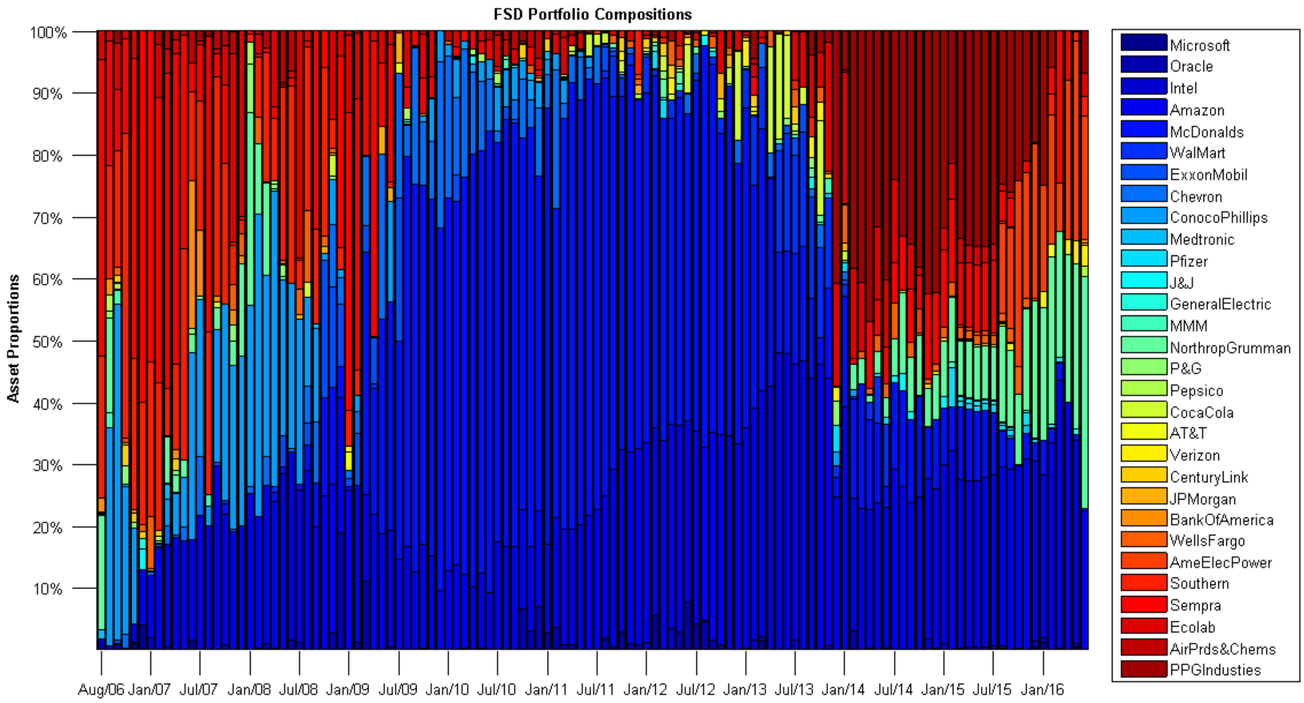

Figure 4.6: Composition of the FSD-Kuosmanen portfolio during the backtesting period.

Figure 4.7 depicts the composition of the SSD-Kuosmanen optimal portfolio. An investor choosing this portfolio will be able to invest in 20 of the total 30 available assets. Specifically the top ten assets are the following: Amazon, McDonalds, ExxonMobil, ConocoPhillips, Northrop-Grumman, American-Electric-Power, Southern, Sempra, Ecolab and PPG Industries, where Amazon and McDonalds provide the highest proportion of wealth to invest in. Finally *Figure 4.8* shows the composition of the TSD-Kuosmanen optimal portfolio. This portfolio consists of a total number of 19 assets among the available 30 assets. Specifically the top ten assets that an investor with this portofolio chooses to hold are the following: Amazon, McDonalds, Wal-Mart, ExxonMobil, ConocoPhillips, Northrop-Grumman, American-Electric-Power, Southern, Sempra and PPG Industries, where Amazon and McDonalds provide the highest proportion of wealth to invest in, as in the previous SSD-Kuosmanen portfolio.

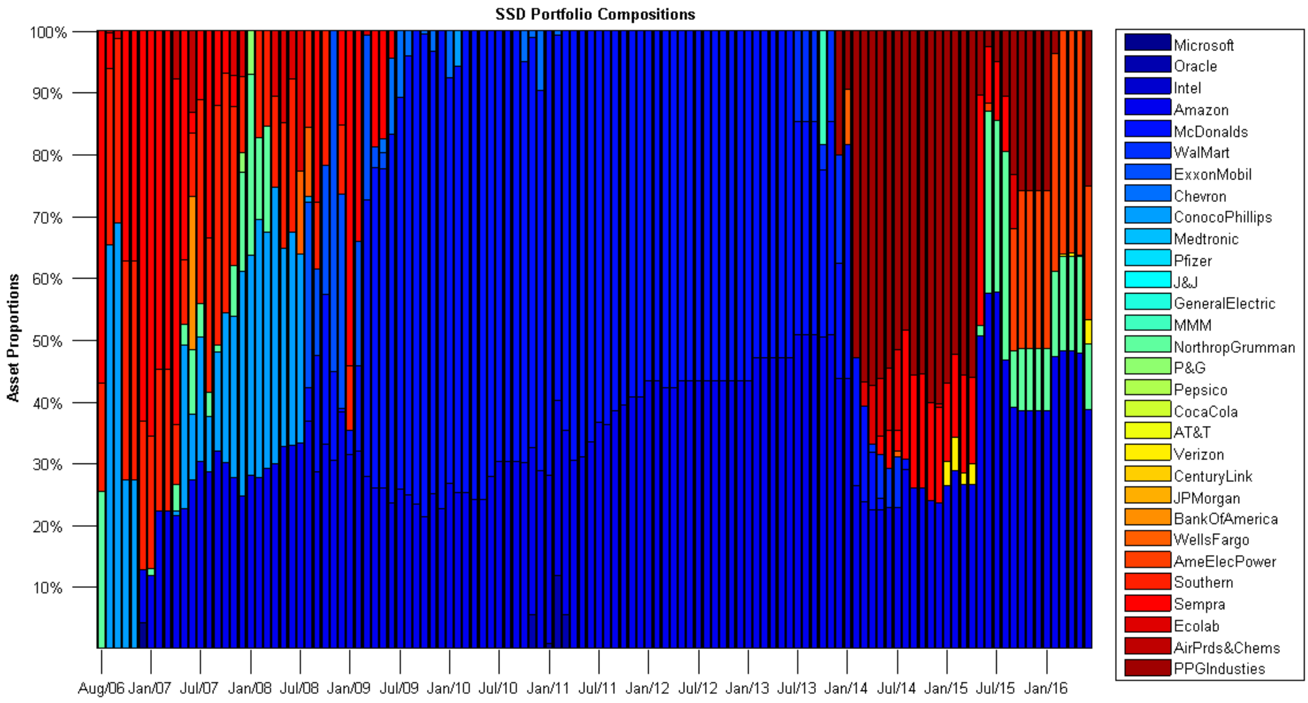

Figure 4.7: Composition of the SSD-Kuosmanen portfolio during the backtesting period.

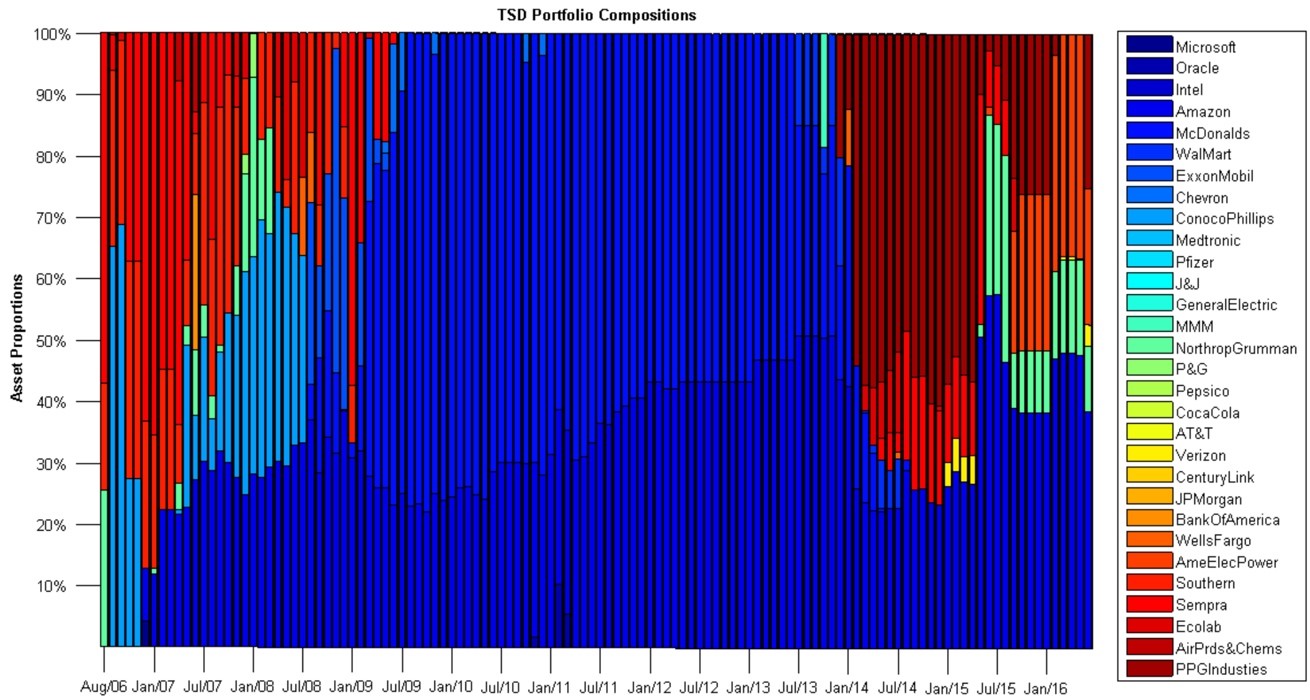

Figure 4.8: Composition of the TSD-Kuosmanen portfolio during the backtesting period.

Chapter 5

Performance Measures

5.1 Introduction

In order to evaluate the performance of the competing portfolios, we compute several standard descriptive statistics of portfolio performance. In particular, we compute mean returns, standard deviation, skewness and kurtosis, as well as minimum and maximum returns over the sample period.

Skewness is a measure of asymmetry that uses the ratio of the average cubed deviations from the average, called the third moment, to the cubed standard deviation to measure any asymmetry of a distribution. In order to estimate the skewness we employ the following formula:

$$Skew = Average\left[\frac{(R-\bar{R})^3}{\hat{\sigma}^3}\right] \qquad (5.1)$$

Cubing deviations maintains their sign. Thus, if a distribution is “skewed to the right”, the extreme positive values, when cubed, will dominate the third moment, resulting in a positive measure of skew. On the other hand, if a distribution is “skewed to the left” the cubed extreme negative values will dominate, and the skew will be negative. When the distribution is positively skewed, the standard deviation overestimates risk, because extreme positive deviations increase the estimated volatility. Conversely, when the distribution is negatively skewed, the standard deviation will underestimate risk.

Kurtosis is a measure of the degree of “fat tails”. In this case, we make use of the deviations from the average raised to the fourth power and standardize by dividing by the fourth power of the standard deviation, that is:

$$Kurtosis = Average\left[\frac{(R-\bar{R})^4}{\hat{\sigma}^4}\right] \qquad (5.2)$$

When the tails of a distribution are “fat”, there is more probability mass in the tails of the distribution than predicted by the normal distribution, at the expense of “slender shoulders”, that is less probability mass near the center of the distribution. The kurtosis of a normal distribution is defined as 3, and any kurtosis above 3 is a sign of fatter tails than would be observed in a normal distribution. Higher frequency of extreme negative returns may result from negative skew or kurtosis below 3.

We report the Sharpe ratio and the upside potential and downside risk ratio UP_{ratio} proposed by Sortino and van der Meer(1991) for each of the portfolios using the one-month US T-bill rate as the risk-free rate. We also include several additional performance measures such as portfolio turnover and opportunity cost.
The Sharpe ratio is the average return earned in excess of the risk-free rate per unit of volatility or total risk. The upside potential and downside risk ratio contrasts the upside potential against a specific benchmark with the shortfall risk against the same benchmark, as suggested by Sortino et al. (1999). Let r_t be the return of a portfolio in month t=1,2,….,k of the simulation; k=120 is the number of months in the simulation period 08/2006-07/2016. Let ρ_t be the risk-free rate at the same period. In order to compute these two ratios, we use the following formulas.

$$Sharpe\ Ratio = \frac{\frac{1}{k}\sum_{t=1}^{k}(r_t-\rho_t)}{\sqrt{\frac{1}{k-1}\sum_{t=1}^{k}(r_t-\bar{r})^2}} \qquad (5.3)$$

$$UP_{ratio} = \frac{\frac{1}{k}\sum_{t=1}^{k}\max\,[0, \mathrm{r_t}-\rho_t]}{\sqrt{\frac{1}{k}\sum_{t=1}^{k}(\max[0,\rho_t-r_t])^2}} \qquad (5.4)$$

where $\bar{r}$ is the average mean return. The numerator of UP_{ratio} is the average excess return compared to the benchmark, reflecting the upside potential.The denominator is a measure of downside risk and can be thought of as the risk of failing to meet the Benchmark.

The portfolio turnover (PT) is calculated as a measure of exposure to transaction costs related with portfolio rebalancing. It is computed as the average absolute change summed across all N portfolio weights:

$$PT = \frac{1}{k}\sum_{t=1}^{k-1}\sum_{i=1}^{N}\left(\left|w_{i,t+1} - w_{i,t}\right|\right) \qquad (5.5)$$

where $w_{i,t+1}$, $w_{i,t}$ are the derived optimal weights of the asset i at time $t+1$ and t respectively. The PT quantity can be interpreted as the average fraction of the portfolio value that has to be reallocated over the entire planning horizon.

Finally, we use the idea of opportunity cost (Simaan, 1993) to determine the economic importance of the difference in performance of the optimal portfolios compared to the Benchmark. Let r_p be the optimal portfolio realized returns obtained by the different strategies and r_{bench} be the returns of the Benchmark. The opportunity cost θ is defined to be the return that should be added to the return r_{bench} so that the investor becomes indifferent between the alternative strategies.

$$E[U(1 + r_{bench} + \theta)] = E\left[U\left(1 + r_p\right)\right] \qquad (5.6)$$

For the previous reason a positive opportunity cost indicates that the investor is in a more advantageous position in the case of pursuing a particular investment strategy. In order to compute the opportunity cost, we use an exponential utility function, with different degrees of absolute risk aversion $(ARA = 2{,}4{,}6)$ and a power utility function, with different degrees of relative risk aversion $(RRA = 2{,}4{,}6)$ alternatively.

The exponential utility function is defined as:

$$U(x) = -\frac{e^{nx}}{n}, n > 0 \qquad (5.7)$$

where n is the coefficient of absolute risk aversion.

The power utility function is defined as:

$$U(x) = \frac{x^{1-\gamma} - 1}{1 - \gamma}, \gamma \neq 1 \qquad (5.8)$$

where γ is the coefficient of relative risk aversion.

5.2 Out-of-sample portfolios: Descriptive statistics of the returns

Table 5.1 reports descriptive statistics for the returns delivered by the alternative portfolio strategies as well as the Benchmark. The three $CVaR$-related strategies have higher mean returns than the Benchmark (0.86% to 3.27% versus 0.70%). Aggressive delivers the highest mean return 3.27%, but the associated costs are the highest standard deviation among all the alternative investment tactics as well as the worst minimum return of -25.40% compared to -16.80% for Bench,-11.17% for Average,-9.28% for Defensive,-11.92% compared to the TSD-Kuosmanen,-12.05% for SSD-Kuosmanen and -15.30% compared to the FSD-Kuosmanen. Moreover it has positive skewness (0.7884) which indicates a return distribution with an asymmetric tail extending toward more positive values and achieves the best maximum monthly return of 54.10%. Kurtosis measures the degree to which a distribution is more or less peaked than a normal distribution. Aggressive is characterized by high kurtosis (6.9326), will have “fat tails” (higher frequencies of outcomes) at the extreme negative and positive ends of its return distribution curve. Average reduces somewhat the standard deviation as compared to Benchmark (4.25% versus 4.41%), while also allowing relatively large gains (maximum return of 14.31%). Furthermore it has negative skewness (-0.1777), so the return distribution is approximately symmetric.

Defensive managed to reduce standard deviation even further (2.96%), however this also lowers its mean return. It achieves the best minimum returns (-9.28% versus -16.80%) and it has negative skewness (-0.0139) which indicates a return distribution with an asymmetric tail extending toward more negative values.

Turning to the Kuosmanen-related strategies, the TSD-Kuosmanen optimal portfolio performs well along multiple dimensions. In particular this portfolio exhibits higher mean return than the Benchmark (2% versus 0.70%), but on the other hand it also yields higher standard deviation (4.70% versus 4.41%). Additionally it reduces the maximum loss (minimum monthly return of -11.92%), while also allowing large gains (maximum return of 17.38%). The TSD-Kuosmanen is characterized by the highest positive value of skewness among all the alternative competing portfolios (0.1359), whereas also generates leptokurtic distribution (3.9274). The SSD-Kuosmanen optimal portfolio improves the mean return (2.03% versus 0.70%), but also has higher standard deviation than the Benchmark (4.69% versus 4.41%). Moreover this portfolio appears to have compatible minimum monthly return and maximum monthly return with the TSD-Kuosmanen optimal portfolio; that is of -12.05% and 17.38%, respectively. This may occur due to the fact that this approach tends to construct a portfolio that will avoid large negative returns and it is characterized by positive skewness (0.1265), meaning frequent small losses and a few extreme gains. SSD-Kuosmanen exhibits leptokurtic distribution (3.9398) which means that its return distribution also has ”fatter” tails and that there are more chances of extreme outcomes compared to a normal distribution.

FSD-Kuosmanen portfolio performs better than the Benchmark with higher mean return (1.79% versus 0.70%) and slightly lower standard deviation (4.35% versus 4.41%). Additionally, it reduces the maximum loss somewhat compared to the Bench (minimum monthly returns of -15.30% versus -16.80%), as well as its maximum return is comparable with the one of the Average portfolio. Finally it has negative skewness (-0.4216), its return distribution is approximately symmetric compared with the return distribution of the Benchmark which is moderately skewed (-0.7300).

	Mean	Standard Deviation	Min	Max	Kurtosis	Skewness
Benchmark	0.70	4.41	-16.80	10.93	4.5205	-0.7300
Aggressive	3.27	10.45	-25.40	54.10	6.9326	0.7884
Defensive	0.86	2.96	-9.28	10.18	3.7805	-0.0139
Average	1.58	4.25	-11.17	14.31	3.8098	-0.1777
TSD-Kuosmanen	2	4.70	-11.92	17.38	3.9274	0.1359
SSD-Kuosmanen	2.03	4.69	-12.05	17.38	3.9398	0.1265
FSD-Kuosmanen	1.79	4.35	-15.30	14.67	4.5746	-0.4216

Table 5.1: Descriptive statistics of portfolio performance: This table reports descriptive statistics for the portfolios chosen by competing strategies based on monthly percentage returns. The statistics are computed using 120 monthly returns from 2006 to 2016.

5.3 Out-of-sample portfolios: Portfolio Performance and Risk Measures

In this section we discuss the results on the out-of-sample performance of the alternative optimal portfolios and the benchmark, using the four standard measures of the performance: the Sharpe ratio, the Sortino ratio, portfolio turnover and the opportunity cost. In the case of opportunity cost, we assume various levels of risk aversion ($ARA, RRA = 2{,}4{,}6$) for the investor. *Table 5.2* reports results for each one of the four performance measures.

We can observe that compared to the Benchmark all the optimal portfolios formed based on different strategies yield greater Sharpe ratio as well as Sortino ratio. Regarding the opportunity cost we can see that it is positive in all cases, which implies that the investor is better off when any of the alternative investment tactics is considered.

The three $CVaR$-related strategies outperform the Benchmark on all the traditional performance measures we examine except turnover, where Bench by definition has none. However, these approaches do not perform as well as Kuosmanen-related strategies in terms of Sharpe and Sortino ratios. Aggressive delivers higher Sharpe ratio and Sortino ratio than the Benchmark. Its turnover value of 5% is the smallest among all the strategies tested, which is not surprising, if we take into account the fact that this investor chooses only one asset during the whole planning horizon. Regarding the opportunity cost, we can observe that this has a positive sign, indicating that the investor is demanding a premium in order to replace that optimal strategy with the Benchmark. Interestingly, in this approach the opportunity cost decreases as the risk aversion increases. This implies that the investor has a tendency to becoming indifferent in utility terms between the two strategies as he becomes more risk averse.

Average has the best Sharpe ratio within the $CVaR$-related group of portfolios and it has quite similar Sortino ratio to Aggressive. Moreover, it also has the largest turnover measure among the $CVaR$-related strategies, indicating that the portfolio weights are not quite stable. Regarding the opportunity cost, we can see that this is positive and it yields higher values as the risk aversion increases.

Due to its basic characteristic of risk avoidance, Defensive appears to have weak performances with the Sharpe and Sortino ratios being 0.2677 and 0.9441, respectively. This generally weaker performance among the $CVaR$-related strategies appears as long as this approach corresponds to the minimum risk case. Only for turnover does Defensive achieve to outperform Average.

Turning to the Kuosmanen-related strategies, we can observe that TSD-Kuosmanen, SSD-Kuosmanen as well as FSD-Kuosmanen outperform the $CVaR$-related strategies (and Benchmark) on all the traditional out-of-sample performance measures we examined except turnover.

Table 5.2: Out-of-sample performance of the alternative portfolios

Entries report the performance measures (Sharpe Ratio, Sortino Ratio, Portfolio Turnover and Opportunity Cost). The dataset for all assets spans the period from January 2000- July 2016. The out-of-sample analysis is conducted over the period from August 2006- July 2016.

	Sharpe ratio	Sortino ratio	Turnover (%)	Opportunity cost (%) *Exponential Utility* $ARA = 2$	Opportunity cost (%) *Exponential Utility* $ARA = 4$	Opportunity cost (%) *Exponential Utility* $ARA = 6$	Opportunity cost (%) *Power Utility* $RRA = 2$	Opportunity cost (%) *Power Utility* $RRA = 4$	Opportunity cost (%) *Power Utility* $RRA = 6$
Benchmark	0.1444	0.6518	0	0	0	0	0	0	0
Aggressive	0.3064	1.0638	5	1.72	0.95	0.19	1.73	0.93	0.13
Average	0.3548	1.0569	29.42	0.89	0.91	0.94	0.87	0.90	0.94
Defensive	0.2677	0.9441	11.71	0.27	0.38	0.51	0.25	0.37	0.51
TSD-Kuosmanen	0.4108	1.2605	23.95	1.27	1.26	1.26	1.26	1.25	1.25
SSD-Kuosmanen	0.4176	1.1996	24.51	1.30	1.29	1.29	1.28	1.28	1.29
FSD-Kuosmanen	0.3958	1.1172	43.14	1.09	1.10	1.12	1.07	1.09	1.11

Compared to the Benchmark, TSD-Kuosmanen optimal portfolio provides higher realized values for the Sharpe ratio and Sortino ratio. In particular, this portfolio delivers the best value for the Sortino ratio among not only the Kuosmanen-related strategies but also among all the alternative competing portfolios. Furthermore its portofolio turnover value is comparable with the SSD-Kuosmanen optimal portfolio. Regarding the opportunity cost, we again see that this has a positive sign and it decreases as the risk aversion increases. This signifies that the investor tends to become indifferent in utility terms between choosing to invest in the Benchmark or in the TSD-Kuosmanen portfolio as he becomes more risk averse.

SSD-Kuosmanen portfolio exhibits the highest realized value for the Sharpe ratio, while at the same time outperforms the $CVaR$-related strategies in terms of Sortino ratio. Moreover its turnover value is considerably lower than Average, which indicates a quite stable portfolio. Considering the opportunity cost, we can observe that this has a positive sign and it provides lower values as the risk aversion increases, as in the case of the TSD-Kuosmanen.

FSD-Kuosmanen portfolio performs better on measures such as the Sharpe and Sortino ratios that give more weight to mean returns, compared to the Benchmark. Furthermore, it achieves to outperform the $CVaR$-related strategies on the traditional performance measures nevertheless it has the worst turnover measure among all the alternative optimal portfolios, indicating that the portfolio weights are not quite stable. Finally, we can see that the opportunity cost is again positive which implies that the investor is better off when this particular investment strategy is considered. In contrast to the previous two portfolios, opportunity cost increases as the risk aversion increases. In our tests, the TSD-Kuosmanen, the SSD-Kuosmanen strategies as well as Average look promising for creating portfolios that perform well out-of-sample.

Chapter 6

Conclusions

In this dissertation,we use the concepts of first-order stochastic dominance, second-order stochastic dominance and third-order stochastic dominance as well as the $CVaR$ approach, including three different investment tactics, in order to construct optimal portfolios. In particular, we propose to determine the optimal portfolios based on the FSD, SSD and FSD criteria to find the optimal portfolio weights. We implement all the alternative models in the General Algebraic Modeling System (GAMS). In constructing our *FSD-based* portfolio we adopt 0-1 Mixed Integer Linear Programming developed in Kuosmanen (2004), as well as the construction of our *SSD-based* and *TSD-based* portfolios are formulated in terms of standard Linear Programming developed once again in Kuosmanen (2001,2004). Furthermore, in order to compare the performance of the optimal competing portfolios, we evaluate all these alternative portfolios with respect to the market benchmark portfolio using several performance measures such as the Sharpe Ratio, the Sortino Ratio, the opportunity cost and portfolio turnover.

In the empirical tests we consider investments in the US market. We want to construct several optimal portfolios based on alternative strategies. We use data on monthly closing prices of S&P500, as well as a number of stocks obtained by Datastream. Considering the $CVaR$ approach, we first maximize the $CVaR$ without any constraint for the defensive investor, due to the fact that this strategy corresponds to the minimum risk case. The next model we examined was the aggressive investor in which we maximized the expected return again without any constraint; in this strategy we want to gain the maximum benefit without any consideration on risk value. The last strategy examined in this particular optimization model was the average investor model; to succeed this we maximized $CVaR$ subject to the expected target return.

Turning to the Kuosmanen-related strategies, we can think of the Stochastic Dominance concepts as properties of the probability distributions. In constructing our FSD-based, SSD-based and TSD-based portfolios, we have already mentioned that we adopt the algorithms developed in Kuosmanen. This approach allows for testing if a benchmark portfolio return distribution is FSD, SSD and TSD efficient respectively, relative to a given asset span. If the benchmark is not efficient, the solution also delivers a vector of portfolio weights corresponding to a well-diversified FSD-efficient, SSD-efficient and TSD-efficient portfolio that first-order, second-order, as well as third-order, stochastically dominates the benchmark. In Kuosmanen, the procedure is developed on the necessary condition for Stochastic Dominance efficiency; and it measures the degree of inefficiency for the benchmark portfolio in terms of FSD, SSD and TSD respectively.

Considering the ex-post realized performance of all the alternative competing portfolios, over the backtesting simulation period, the minimum risk portfolio (i.e. Defensive-portfolio) provides a stable growth path over the planning horizon without any significant levels of volatility. Moreover, the Average-portfolio achieves quite stable growth paths, with slight losses in only few instances during the simulation period. Hence,the Average portfolio yields a clearly superior performance compared to the Defensive-portfolio. Regarding the Aggressive portfolio, this particular investment strategy shows a noticeable improvement in performance; although this approach exhibits the highest fluctuations in returns, reflecting a riskier portfolio, compared to the previous two tactics, it results in higher cumulative returns. The FSD-Kuosmanen optimal portfolio exhibits stable portfolio returns throughout the planning horizon period, with small losses in only very few periods, as well as its realized return paths are discernibly more stable that the corresponding path of the optimal Aggressive-portfolio. As for the SSD-Kuosmanen optimal portfolio and the TSD-Kuosmanen optimal portfolio, both these two strategies demonstrate very similar ex post performance; with the SSD-Kuosmanen portfolio being a very slight favorite. We can clearly state that these two tactics exhibit superior performance compared not only to the Average-portfolio and to the Defensive-portfolio, but also compared to the Benchmark.

Regarding the statistical characteristics of the competing portfolios, the three $CVaR$-related strategies have higher mean returns than the Benchmark. Moreover, Aggressive delivers the highest mean return, but the associated cost is the highest standard deviation among all the alternative investment tactics. FSD-Kuosmanen portfolio performs better than the Benchmark with higher mean return and slightly lower standard deviation. The TSD-Kuosmanen optimal portfolio, as well as the SSD-Kuosmanen portfolio provide quite similar mean returns and standard deviations, while at the same time they exhibit higher mean return than the Benchmark but on the other hand they also yield higher standard deviation.

Finally with regard to the performance measures, the three $CVaR$-related strategies outperform the Benchmark on all the traditional performance measures we examine except turnover, where Bench by definition has none. However, these approaches do not perform as well as Kuosmanen-related strategies in terms of Sharpe and Sortino ratios. Considering the opportunity cost we can see that it is positive in all cases, which implies that the investor is better off when any of the alternative investment tactics is considered.

SSD-Kuosmanen portfolio exhibits the highest realized value for the Sharpe ratio, while at the same time outperforms the $CVaR$-related strategies in terms of Sortino ratio. Moreover its turnover value is considerably lower than Average, which indicates a quite stable portfolio. Compared to the Benchmark, TSD-Kuosmanen optimal portfolio provides higher realized values for the Sharpe ratio and Sortino ratio. In particular, this portfolio delivers the best value for the Sortino ratio among not only the Kuosmanen-related strategies but also among all the alternative competing portfolios. Furthermore its portofolio turnover value is comparable with the SSD-Kuosmanen optimal portfolio. FSD-Kuosmanen portfolio performs better on measures such as the Sharpe and Sortino ratios that give more weight to mean returns, compared to the Benchmark, but it also has the worst turnover measure among all the alternative optimal portfolios, indicating that the portfolio weights are not quite stable.

In our tests, we can conclude that in the backtesting simulations the TSD-Kuosmanen as well as the SSD-Kuosmanen strategies look promising for creating portfolios that perform well out-of-sample, as they provide diversified and stable portfolios with higher resilience during market downturns, and lower turnover compared to the FSD-Kuosmanen portfolio. Finally, with regard to the $CVaR$-related strategies, we can state that the Average portfolio produces more effective ex post realized return paths, compared to the Benchmark, both in terms of higher growth rates and best values concerning the performance measures.

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