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Abstract

This paper proposes a framework for comparing the predictions of representative household models with those of models of overlapping generations, in the context of a class of endogenous growth theories with investment adjustment costs. In the model used in this paper, savings and investment are co-determined through adjustments in the real interest rate, and the equilibrium investment rate determines the long-run growth rate. The two classes of models have similar predictions regarding the effects of technological and preference shocks, but the overlapping generations model predicts lower savings and investment, higher interest rates and lower growth rates that the corresponding representative household model. We calibrate the two models using similar parameter values and the results suggest that the differences between the two models are not quantitatively large. For plausible parameter values, the differences in growth rates, savings rates and investment rates are of the order of 0.1 to 0.2 of a percentage point per annum, which accumulated over twenty five years is at most 5% of aggregate output. The differences for real interest rates are even smaller. Overall the results suggest that the relative simplicity of the representative household model does not lead to results that would be too far off quantitatively, even if the real world is characterized by overlapping generations.

Keywords: savings, investment, endogenous growth, interest rates, representative household, overlapping generations
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The Solow (1956) neoclassical growth model is the starting point of most analyses of long run growth. Its main prediction is that the savings rate determines the rate of accumulation of physical capital. However, the Solow model does not consider optimizing households and it treats the savings rate as exogenous. In addition, it has no separate investment theory, as aggregate investment is simply determined by aggregate savings.

There are two competing intertemporal theories of the determination of aggregate savings in the context of the neoclassical growth model. The Ramsey (1928)-Cass (1965)-Koopmans (1965) representative household model, and the Diamond (1965)-Blanchard (1985)-Weil (1989) models of overlapping generations. In both classes of models, households engage in intertemporal optimization and their savings behavior is individually optimal. However, in the overlapping generations model, the social outcome is suboptimal, in the sense that optimizing households do not take into account the welfare of future generations. As a result, savings are usually sub-optimally low.

There are a number of other differences between the representative household and the overlapping generations theories of the determination of aggregate savings. Representative household models are characterized by both the neutrality of government debt (Ricardian equivalence) and the super-neutrality of money. Neither of the two hold in overlapping generations models, in which both government debt and monetary growth are not neutral and can have real effects.[1]

However, apart from their differences, representative household and overlapping generations models, have at least two features in common with the Solow neoclassical growth model. First, there is no separate investment theory, and, second, the long run growth rate is exogenous and independent of savings behavior.

The most prevalent investment theory utilized in the literature is the *q* theory of investment (see Lucas 1967, Gould 1968, Tobin 1969, Abel 1982, Hayashi 1982). In this theory, it is assumed that firms face convex internal costs to adjusting their capital stock. Optimal investment by competitive firms thus depends on Tobin's *q*, the ratio of the market value of installed capital to the replacement cost of capital. The *q* theory of investment is an investment theory separate from the alternative savings theories. In equilibrium, investment is not determined by aggregate savings, but savings and investment are co-determined in competitive capital markets.

Irrespective of their differences, all models that rely on the assumptions of the Solow neoclassical growth model cannot escape from a key characteristic of this model. The accumulation of physical capital cannot account for either the vast growth over time of output per person, or the vast geographic differences in output per person between countries, regions in different parts of the world. The differences in real incomes that we are trying to understand are far too large to be accounted for by differences in capital inputs, based on the assumption of competitive markets and a standard neoclassical production function. Models based on externalities (Arrow 1962), increasing returns (Romer 1986, 1990) and human capital accumulation (Lucas 1988) account far better for these differences. Chief among them are the endogenous growth models of Romer and Lucas.[2]

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[1] See for example van der Ploeg and Alogoskoufis (1994).

[2] See Romer (1986), Lucas (1988), and a number of comprehensive recent surveys such as Barro and Sala-i-Martin (2004), Aghion and Hewitt (2009), Acemoglou (2009).

This paper proposes a framework for comparing and assessing the macroeconomic predictions of representative household models with those of models of overlapping generations. The model used in this paper belongs to a class of endogenous growth theories with investment adjustment costs, in which savings and investment are co-determined through adjustments in the real interest rate. Due to external effects of capital accumulation, the investment rate determines the long-run growth rate. The two classes of models have similar predictions regarding the effects of technological and preference shocks, but the overlapping generations model predicts lower savings and investment, higher interest rates and lower growth rates that the corresponding representative household model. We calibrate the two models using similar parameter values and the results suggest that the differences between the two models are not quantitatively large. For plausible parameter values, the differences in growth rates, savings rates and investment rates are of the order of 0.1 to 0.2 of a percentage point per annum, which accumulated over twenty five years is at most 5% of output. The differences for real interest rates are even smaller. Overall the results suggest that the relative simplicity of the representative household model does not lead to results that would be too far off quantitatively, even if the real world is characterized by overlapping generations.

The rest of the paper is as follows: Section 1 uses a *q* model of investment, with convex adjustment costs, to characterize the investment decisions of firms. We assume learning by doing and constant returns to the accumulation of physical capital. In section 2 we model the relationship between the endogenous growth rate and the real interest rate, that is required for equilibrium investment. We show that the endogenous growth rate depends negatively on the real interest rate, and positively on the productivity of capital. In sections 3, 4 and 5 we address savings behavior. In section 3, we use the simple Solow assumption of an exogenous savings rate to show how an increase in the savings rate reduces the real interest rate and results in an increase in the investment rate and the growth rate. In sections 4 and 5 we use a representative household and an overlapping generations model respectively. In both the representative household model and the overlapping generations model, the endogenous growth rate and the real interest rate are co-determined through the interaction of equilibrium investment by firms and equilibrium savings by households. The assumption of adjustment costs for investment is crucial in this respect. Without investment adjustment costs, i.e with ϕ=0, this co-determination does not apply in the endogenous growth model. The production side determines the real interest rate, as the net marginal product of capital, and, given the real interest rate, the consumption side determines the growth rate of consumption, capital and output. In section 6 we present the calibration results and section 7 sums up the conclusions.

1. The Investment Decisions of Firms

We assume an economy, consisting of a large number of competitive firms that produce a single homogeneous good.

2.1 Production

The production function of firm *i* at time *t* is given by,

$$Y_{it} = AK_{it}^{\alpha}(h_t L_{it})^{1-\alpha}, \quad 0<\alpha<1 \qquad (1)$$

where Y is output, K physical capital, L the number of employees and h the efficiency of labour. The efficiency of labour is the same for all firms.

Following Arrow (1962) we assume *learning by doing*. In particular we assume that the efficiency of labour (human capital per worker) is a linear function of the aggregate ratio of physical capital to labor. Thus,

$$h_t = B\left(\frac{K}{L}\right)_t, \quad 0<\beta<1 \tag{2}$$

where B is a constant, and K/L is the aggregate capital labour ratio.

Substituting (2) in (1) and aggregating, we get aggregate output as a linear function of aggregate physical capital.

$$Y_t = \bar{A} K_t \tag{3}$$

where,

$$\bar{A} = AB^{1-\alpha} \tag{4}$$

In what follows we shall refer to $\bar{A}$ as the *aggregate productivity of capital*. Average and marginal aggregate productivity are constant and obviously equal, due to the linearity of the aggregate production function (3).[3]

Due to the linearity of the aggregate production function, the (endogenous) rate of economic growth g will be equal to the rate of net capital accumulation, which is in turn determined by the rate of investment. Therefore, we shall have,

$$g = \dot{Y}_t / Y_t = \dot{K}_t / K_t = (I_t / K_t) - \delta = \bar{A}(I_t / Y_t) - \delta \tag{5}$$

where I is gross investment and δ the rate of depreciation.

In this endogenous growth model, the long run rate of growth is determined by the ratio of gross investment to GDP as well as the productivity of capital. Investment in physical capital is the driving force of the long run growth process, so we now turn to the determination of investment.

2.2. Adjustment Costs and the Rate of Investment

Investment is determined by the profit maximization decisions of private firms. We assume that new investment is subject to a marginal adjustment cost which is a function of the ratio of new

[3] The linearity of the aggregate production function follows from the assumed linearity in the production of human capital (efficiency of labor) in (2). The qualitative implications of this model would be similar to the implications for the transition path in an exogenous growth model.

investment goods to total installed capital (see Lucas 1967, Gould 1968, Abel 1982 and Hayashi 1982).

Thus, the instantaneous profits of firms are given by,

$$Y_{it} - w_t L_{it} - \left[1 + \frac{\phi}{2}\left(\frac{I_{it}}{K_{it}}\right)\right] I_{it} \tag{6}$$

where *w* is the real wage and ϕ is a positive constant measuring the intensity of the marginal adjustment cost of new investment.

$\phi\left(\frac{I_{it}}{K_{it}}\right)$ is the marginal adjustment cost.

Each firm selects employment and investment in order to maximize the present value of its profits.

$$V_{it} = \int_{s=t}^{\infty} e^{-rs}\left(Y_{is} - w_s L_{is} - \left[1 + \frac{\phi}{2}\left(\frac{I_{is}}{K_{is}}\right)\right] I_{is}\right) ds \tag{7}$$

under the constraint,

$$\dot{K}_{is} = I_{is} - \delta K_{is} \tag{8}$$

r is the real domestic interest rate.

From the first order condition for a maximum of (7) subject to (8),

$$w_t = (1-\alpha) A \left(\frac{K_{it}}{L_{it}}\right)^{\alpha} h_t^{1-\alpha} \tag{9}$$

$$q_{it} = 1 + \phi\left(\frac{I_{it}}{K_{it}}\right) = 1 + \phi\left(\frac{\dot{K}_{it}}{K_{it}} + \delta\right) \tag{10}$$

$$\left(r + \delta - \frac{\dot{q}_{it}}{q_{it}}\right) q_{it} = \alpha A \left(\frac{K_{it}}{L_{it}}\right)^{\alpha-1} h_t^{1-\alpha} + \frac{\phi}{2}\left(\frac{\dot{K}_{it}}{K_{it}} + \delta\right)^2 \tag{11}$$

where q_i is the shadow price of installed physical capital (Tobin's *q*) for firm *i*.

The details of deriving the first order conditions (9) to (11) are well known (see Lucas 1967 Gould 1967) and are therefore omitted.

From (9), employment is determined so that the marginal product of labour for the firm equals the real wage. Given that the real wage is the same for all firms, and all firms have the same production function, all firms will choose the same capital-labour ratio.

From (10), the shadow price of installed capital is equal to the marginal cost of new investment. This is equal to the cost of purchase of new capital goods, plus the marginal adjustment cost of investment.

From (11), the *user cost of capital* (on the left hand side) is equal to the augmented marginal product of capital (on the right hand side). The marginal product of capital has two components: The marginal product in terms of additional output (the first term on the right hand side) and the reduction of the adjustment cost of future investment (the second term on the right hand side). A higher capital stock today means a smaller marginal adjustment cost for future investment.

It is worth noting that if there are no adjustment costs for investment, then q is equal to one (from (10)). (11) then becomes the well known condition that the real interest rate r is equal to the net marginal product of capital.

Thus, if $\phi = 0$, which implies $q=1$, then, $r = \alpha A\left(\frac{K_{it}}{L_{it}}\right)^{\alpha-1} h_t^{1-\alpha} - \delta$.

We shall return to this special case below, arguing in favor of adjustment costs for investment on both theoretical and empirical grounds.

2. The Endogenous Growth Rate and the Real Interest Rate

Aggregating (9) to (11), taking into account (2) to (5), we have the following *aggregate first order conditions*.

$$w_t = (1-\alpha)\bar{A}\left(\frac{K_t}{L_t}\right) \tag{12}$$

$$q_t = 1 + \phi\left(\frac{\dot{K}_t}{K_t} + \delta\right) = 1 + \phi\left(g + \delta\right) \tag{13}$$

$$\left(r + \delta - \frac{\dot{q}_t}{q_t}\right) q_t = \alpha\bar{A} + \frac{\phi}{2}\left(\frac{\dot{K}_t}{K_t} + \delta\right)^2 = \alpha\bar{A} + \frac{\phi}{2}\left(g + \delta\right)^2 \tag{14}$$

From (13), the growth rate is a linear function of q, the shadow price of capital.

$$g = \frac{q_t - 1}{\phi} - \delta \tag{15}$$

Solving (14) for q we get,

$$q_t = \frac{1}{r+\delta}\left(\alpha \bar{A} + \frac{\phi}{2}(g+\delta)^2 \right) \tag{16}$$

From (13) and (16) it follows that,

$$(r+\delta)(1+\phi(g+\delta)) = \alpha \bar{A} + \frac{\phi}{2}(g+\delta)^2 \tag{17}$$

Equation (17) is a quadratic equation in g and has two solutions which lie on either side of r, the real interest rate. Only the solution with $g < r$ is stable in the sense of satisfying the transversality condition for the maximization of the present value of profits for firms. This solution implies that the growth rate g consistent with equilibrium investment is determined by,

$$g = r - \sqrt{r^2 - \frac{2}{\phi}\left(\alpha \bar{A} - (r+\delta)(1+\phi\delta) \right) - \delta^2} \tag{18}$$

(18) describes the locus of growth rates and interest rates which are consistent with equilibrium investment by firms. This equilibrium locus is described by the downward sloping curve in Figure 1, which we term *equilibrium investment*. The position of this equilibrium investment schedule depends only on the productivity of domestic capital, the depreciation rate and the adjustment cost parameter.

Equilibrium q, will be determined by substituting (18) in (13).

In what follows we assume that the equilibrium growth rate is real, which requires that,

$$r^2 \geq \frac{2}{\phi}\left(\alpha \bar{A} - (r+\delta)(1+\phi\delta) \right) - \delta^2$$

Under this assumption, one can prove the following two properties of the equilibrium investment locus.

First, *the growth rate depends negatively on the real interest rate*.

Proof : From (18), the first derivative of the endogenous growth rate with respect to the real interest rate is given by,

$$\frac{\partial g}{\partial r} = -\frac{g + \frac{1}{\phi}(1+\phi\delta)}{\sqrt{r^2 - \frac{2}{\phi}\left(\alpha\bar{A} - (r+\delta)(1+\phi\delta)\right) - \delta^2}} < 0$$

It is also straightforward to prove that,

$$\frac{\partial^2 g}{\partial r^2} > 0$$

Second, *the growth rate depends positively on the aggregate productivity of capital*.

Proof : From (18), the first derivative of the the equilibrium growth rate with respect to the aggregate productivity of capital is given by,

$$\frac{\partial g}{\partial \bar{A}} = \frac{\alpha/\phi}{\sqrt{r^2 - \frac{2}{\phi}\left(\alpha\bar{A} - (r+\delta)(1+\phi\delta)\right) - \delta^2}} > 0$$

It is worth noting that the difference between the real interest rate and the endogenous growth rate is given by,

$$r - g = \sqrt{r^2 - \frac{2}{\phi}\left(a\bar{A} - (r+\delta)(1+\phi\delta)\right) - \delta^2} \qquad (19)$$

It is straightforward to show that *the real interest rate endogenous growth differential* is a positive function of the real interest rate and a negative function of the productivity of domestic capital.

Of course, the real interest rate is an endogenous variable. To determine the endogenous growth rate and the real interest rate the aggregate investment function (18) (as depicted in Figure 1) does not suffice. One has to look at the determination of aggregate savings.

3. The Solow Model: An Exogenous Savings Rate

We shall start with the assumption that the savings rate is an exogenous constant, as is the case in the Solow (1956) model. This will allow us to look at the effects of the savings rate on endogenous growth and the real interest rate. We shall then move on to the representative household model and the overlapping generations model.

Aggregate spending is assumed to consist of private consumption plus investment.

$$Y_t = C_t + q_t I_t \qquad (20)$$

We assume that aggregate savings are a constant fraction *s* of output. Thus,

$$Y_t - C_t = sY_t = q_t I_t \tag{21}$$

From (21), (5) and (13), it follows that,

$$s = q_t\left(\frac{I_t}{Y_t}\right) = \frac{1}{\bar{A}}\big(1+\phi(g+\delta)\big)(g+\delta) \tag{22}$$

For a constant s, (22) has two solutions for g, which lie on either side of zero. These are,

$$g_{1,2} = -\frac{1}{2\phi}\left(1 \pm \sqrt{1+4\phi s\bar{A}}\right) - \delta \tag{23}$$

Only one of the solutions results in a positive growth rate. For this positive root, it follows that,

$$\frac{\partial g}{\partial s} = \frac{\bar{A}}{\sqrt{1+4\phi s\bar{A}}} > 0 \tag{24}$$

An increase in the savings rate, results in an increase in the growth rate consistent with equilibrium between savings and investment.

The full equilibrium is depicted in Figure 2. Equilibrium growth is determined by the equality of savings and investment as described in (23). This equilibrium condition is independent of the real interest rate, as the savings rate is assumed exogenous and constant. The real interest rate is determined at the intersection of the savings-investment balance locus (23) with the equilibrium investment locus (18).

The effects of an increase in the savings rate are depicted in Figure 3. A rise in the savings rate causes the savings-investment balance locus to shift upwards. In the new equilibrium, E′, growth is higher. The higher investment required to sustain this higher growth rate requires a fall in the real interest rate. Thus, an increase in the savings rate results in a fall in the real interest rate and an increase in the investment and growth rates.

The assumption of an exogenous savings rate is considered a key weakness of the Solow model, as models of intertemporal optimization by households are not generally consistent with a constant savings rate. We thus, turn to the two key classes of models of intertemporal optimization, the representative household model and the overlapping generations model.

4. The Representative Household Model

We first assume an economy consisting of a large number of infinitely lived, identical households. Household j maximizes,

$$U_j = \int_{s=0}^{\infty} e^{-(\rho-n)s}\ln(c_{js})\,ds \qquad , j = 1,2,...,J \tag{25}$$

where c_j is per capita consumption of household j, ρ is the pure rate of time preference and n is the growth rate of the size of household j. Both the pure rate of time preference and the population growth rate are independent of j. We shall assume that ρ-$n>0$, so that lifetime utility does not diverge.

Utility is maximized subject to the flow budget constraint,

$$\dot{a}_{js} = (r_s - n)a_{js} + w_s - c_{js} \tag{26}$$

and the household's solvency (no-Ponzi game) condition,

$$\lim_{t\to\infty} e^{-\int_{s=0}^{t}(r_s-n)ds} a_{jt} = 0 \tag{27}$$

where a_{js} is per capita non-human wealth of household j at instant s, and w_s is per capita non asset (labor) income of household j at instant s. Assuming that each member of household j supplies one unit of labor, this is equal to the real wage, and is independent of j.

Maximization of (25) subject to (26) and (27) yields an Euler equation for consumption which takes the form,

$$\dot{c}_{js} = (r_s - \rho)c_{js} \tag{28}$$

From (28), aggregate consumption at time t, as a share of total output, will be governed by,

$$\dot{c}_t = (r_t - \rho - g + n)c_t \tag{29}$$

where c_t is the share of aggregate consumption in total output at time t.

In steady state equilibrium, the share of consumption in total output will be constant. This requires that,

$$g = r - \rho + n \tag{30}$$

(30) describes the relationship between the real interest rate and the growth rate that is required for equilibrium savings by households. It is depicted in Figure 4, along with the equilibrium investment schedule, as the (upward sloping) equilibrium savings schedule.

General equilibrium for the representative household endogenous growth model with investment adjustment costs is at the intersection of the equilibrium investment and the equilibrium savings schedule, also depicted in Figure 4.

The endogenous growth rate and the real interest rate are co-determined through the interaction of equilibrium investment by firms and equilibrium savings by households.

The assumption of adjustment costs for investment is crucial in this respect. Without investment adjustment costs, i.e with ϕ=0, this co-determination does not apply in the endogenous growth model. The production side determines the real interest rate, as the net marginal product of capital, and, given the real interest rate, the consumption side determines the growth rate of consumption, capital and output.

This is straightforward to demonstrate. Assume that ϕ=0. Then (17) becomes,

$$r = \alpha \bar{A} - \delta \tag{31}$$

The real interest rate is determined without any reference to the preferences of consumers. It is only determined by the (exogenous) net marginal product of capital.

Substituting (31) in (30), the growth rate is then given by,

$$g = \alpha \bar{A} - \delta - \rho + n \tag{32}$$

Equations (31) and (32) describe the determination of the real interest rate and the growth rate in the absence of adjustment costs for investment. The real interest rate does not depend on household preferences, but only on the exogenous net marginal product of capital.

(18) and (30) describe the co-determination of the real interest rate and the growth rate in the presence of investment adjustment costs. In this case, both technology and the preferences of consumers affect the real interest rate.

Consider a decrease in the pure rate of time preference of consumers. The analysis is in Figure 5. This shifts the equilibrium savings schedule to the left, and in the new equilibrium the real interest rate is lower and the growth rate higher. Without adjustment costs for investment, there would have been no effect on the real interest rate.

In Figure 6 we consider the effects of an increase in the marginal product of capital. This shifts the equilibrium investment schedule to the right, and in the new equilibrium both the real interest rate and the growth rate are higher.

5. The Overlapping Generations Model

We next turn to the overlapping generations model. We shall assume a model in which population growth comes in the form of entry of new households into the economy. Thus households differ by their date of birth. All households are infinitely lived, but each generation is only concerned about its own welfare and not the welfare of forthcoming generations.

nL_t households are born at each instant, where L_t is total population at instant t, and n is the rate of growth of the number of households (and population). We shall further assume that each household supplies one unit of labor. Therefore n is also the rate of growth of the labor force.

The household born at instant j chooses chooses consumption to maximize,

$$U_j = \int_{s=j}^{\infty} e^{-\rho s} \ln(c_{js})\, ds \tag{33}$$

subject to the instantaneous budget constraint,

$$\dot{a}_{js} = r_s a_{js} + w_{js} - c_{js} \tag{34}$$

and the household's solvency (no-Ponzi game) condition,

$$\lim_{t\to\infty} e^{-\int_{s=j}^{t} r_s\, ds} a_{jt} = 0 \tag{35}$$

Maximization of (33) subject to (34) and (35) yields an Euler equation for consumption which takes the form,

$$\dot{c}_{js} = (r_s - \rho) c_{js} \tag{36}$$

We can use the first order condition (36) to derive aggregate consumption. From (36) and the household's present value budget constraint it follows that,

$$c_{js} = \rho\left(a_{js} + \int_{\nu=s}^{\infty} w_{j\nu} e^{-\int_{\psi=s}^{\nu} r_\psi\, d\psi}\, d\nu \right) \tag{37}$$

Consumption is linear in total wealth because of the assumption of the constant elasticity of inter-temporal substitution. Furthermore, as we have assumed that the elasticity of inter-temporal substitution is equal to 1, the propensity to consume out of wealth is independent of the real interest rate and only depends on the pure rate of time preference.

The size of the cohort born at time j equals nL_j, where $L_j=exp(nj)$ is the population size at time j. Population aggregates are defined as,

$$C_t = n \int_{j=-\infty}^{t} c_{jt} e^{nj}\, dj \tag{38}$$

Aggregating over cohorts, assuming that newly born households do not inherit any wealth, yields,

$$\dot{C}_t = (r_t - \rho + n) C_t - n\rho A_t \tag{39}$$

where C is aggregate consumption and A is aggregate household non-human wealth.

Equation (39) determines the evolution of aggregate consumption. Dividing (39) by total output Y, yields,

$$\dot{c}_t = (r_t - g - \rho + n)c_t - n\rho q \bar{A}^{-1} \tag{40}$$

In (40) we have assumed that all household non-human wealth is held in the form of shares in domestic firms. The value of this wealth is equal to qK. It then follows from the aggregate production function (3), that,

$$q_t \frac{K_t}{Y_t} = q_t \bar{A}^{-1} \tag{41}$$

From (40), in steady state equilibrium,

$$c = \frac{n\rho q}{(r - g - \rho + n)\bar{A}} \tag{42}$$

Aggregate savings as a share of output are given by,

$$1 - c = 1 - \frac{n\rho q}{(r - g - \rho + n)\bar{A}} = \frac{(r - g - \rho + n)\bar{A} - n\rho\left(1 + \phi(g + \delta)\right)}{(r - g - \rho + n)\bar{A}} \tag{43}$$

In (43) we have used (13) to substitute for q.

Equation (43) describes the relationship between the real interest rate and the growth rate that is required for equilibrium savings by households. It is the equivalent of (30) for the representative household model.

The slope of the equilibrium savings locus is given by,

$$0 < \frac{dg}{dr} = \frac{n\rho\left(1 + \phi(g + \delta)\right)}{n\rho\left(1 + \phi(g + \delta)\right) + n\rho\phi(r - g - \rho + n)} < 1 \tag{44}$$

The equilibrium savings locus has a positive slope. However, the slope is lower than one, which is the slope of the comparable representative household model. In addition it is straightforward to show that the slope is declining in the real interest rate. The equilibrium savings locus lies to the right of the equilibrium savings locus of the comparable representative household model, as for any growth rate, the real interest rate is higher.

The equilibrium savings locus (43) is depicted in Figure 7, along with the equilibrium savings schedule of the comparable representative household model. It is easy to see that the real interest rate is higher and the growth rate is lower in the overlapping generations (OLG) case than in the comparable representative household (RH) model.

As in the representative household model, in the overlapping generations model, the endogenous growth rate and the real interest rate are co-determined through the interaction of equilibrium investment by firms and equilibrium savings by households.

The assumption of adjustment costs for investment is again crucial in this respect. Without investment adjustment costs, i.e with ϕ=0, this co-determination does not apply in the endogenous growth model. The production side determines the real interest rate, as the net marginal product of capital, and, given the real interest rate, the consumption side determines the growth rate of consumption, capital and output.

6. Representative Households versus Overlapping Generations

We have seen that the overlapping generations model implies a distortion that results in higher real interest rates and lower endogenous growth rates. This distortion arises from the fact that current households do not provide for the welfare of future generations. How important is this distortion; This is the question to which we now turn.

In order to assess the magnitude of this distortion, we calibrate the two alternative models, using a common set of parameter values. We assume that the share of labor in the production function is equal to one third (α=0.33), that the capital output ratio is equal to 3 (aggregate productivity of capital equal to 0.33), that the depreciation rate is equal to 1%, the population growth rate is equal to 1% and that the adjustment cost parameter is equal to 50. The results with the pure rate of time preference ranging from 2% to 4% are presented in Tables 1 and 2.

As can be seen by comparing the two tables, the differences between the representative household model and the overlapping generations model are quantitatively small. As predicted by the theory, the overlapping generations model results in higher consumption, higher interest rates and lower growth rates. However, the differences are of the order of 0.2-0.1 of a percentage point for the aggregate growth rate and the savings and investment rates, and even smaller for the real interest rate. A difference of 0.2 of a percentage point in the growth rate, accumulated over 25 years, would result in a output which is higher by about 5%. This is not insignificant, but economies often experience larger output losses during recessions.

Overall, it appears that the much simpler representative household model would not be too far off quantitatively, even if the world is characterized by overlapping generations.

7. Conclusions

This paper compares the predictions of representative household models with those of models of overlapping generations, in the context of a class of endogenous growth theories with investment adjustment costs.

In the model used in this paper, savings and investment are co-determined through adjustments in the real interest rate, and the equilibrium investment rate determines the long-run growth rate. The two classes of models have similar predictions regarding the effects of technological and preference shocks, but the overlapping generations model results in lower savings and investment, higher interest rates and lower growth rates that the corresponding representative household model.

We calibrate the two models using similar parameter values and the results suggest that the differences between the two models are not quantitatively large. For plausible parameter values, the differences in growth rates, savings rates and investment rates are of the order of 0.1 to 0.2 of a percentage point per annum, which accumulated over twenty five years is about 5%. The differences for real interest rates are even smaller. Overall the results suggest that the relative simplicity of the representative household model does not lead to results that would be too far off quantitatively, even if the world is characterized by overlapping generations. The framework used in this paper can be extended to assess other differences between the representative household and overlapping generations models, such as the question of Ricardian equivalence.

Table 1
Calibration of the Representative Household Model

ρ	c	r	g
2.0%	96.1%	4.0%	3.0%
2.5%	96.7%	4.1%	2.6%
3.0%	97.3%	4.2%	2.2%
3.5%	97.7%	4.3%	1.8%
4.0%	98.1%	4.5%	1.5%

Note: We assume that α=0.33, $\bar{A}$=0.33, ϕ=50, δ=1% and n=1%. The calculations have been performed using the Wolfram Alpha computational knowledge engine.
See http://www.wolframalpha.com .

Table 2
Calibration of the Overlapping Generations Model

ρ	c	r	g
2.0%	96.3%	4.0%	2.8%
2.5%	97.0%	4.1%	2.4%
3.0%	97.5%	4.3%	2.0%
3.5%	97.9%	4.4%	1.7%
4.0%	98.3%	4.6%	1.4%

Note: We assume that α=0.33, $\bar{A}$=0.33, ϕ=50, δ=1% and n=1%. The calculations have been performed using the Wolfram Alpha computational knowledge engine.
See http://www.wolframalpha.com .

Figure 1
Adjustment Costs and the Equilibrium Investment Schedule

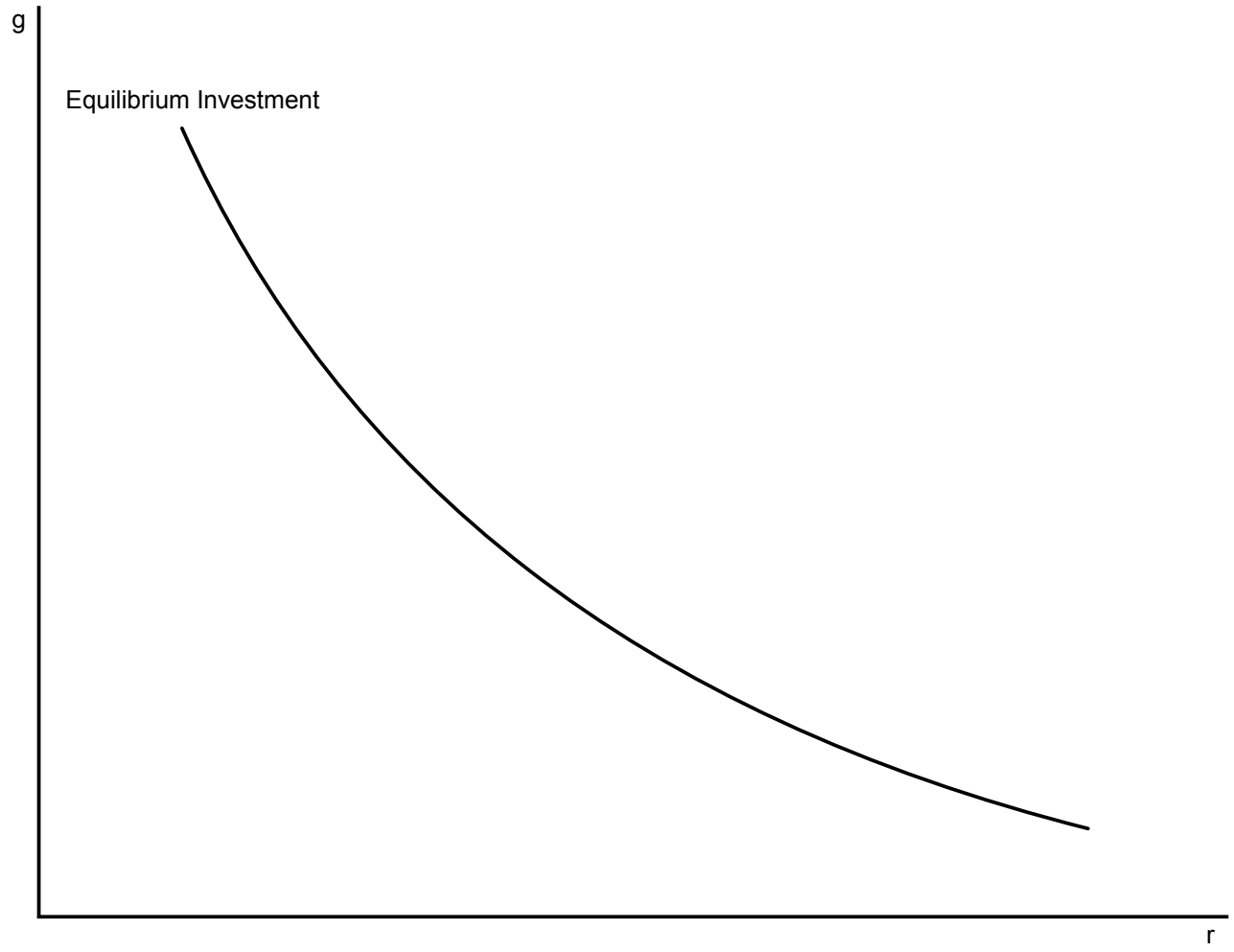


Figure 2
Equilibrium Growth and the Real Interest Rate with an Exogenous Savings Rate

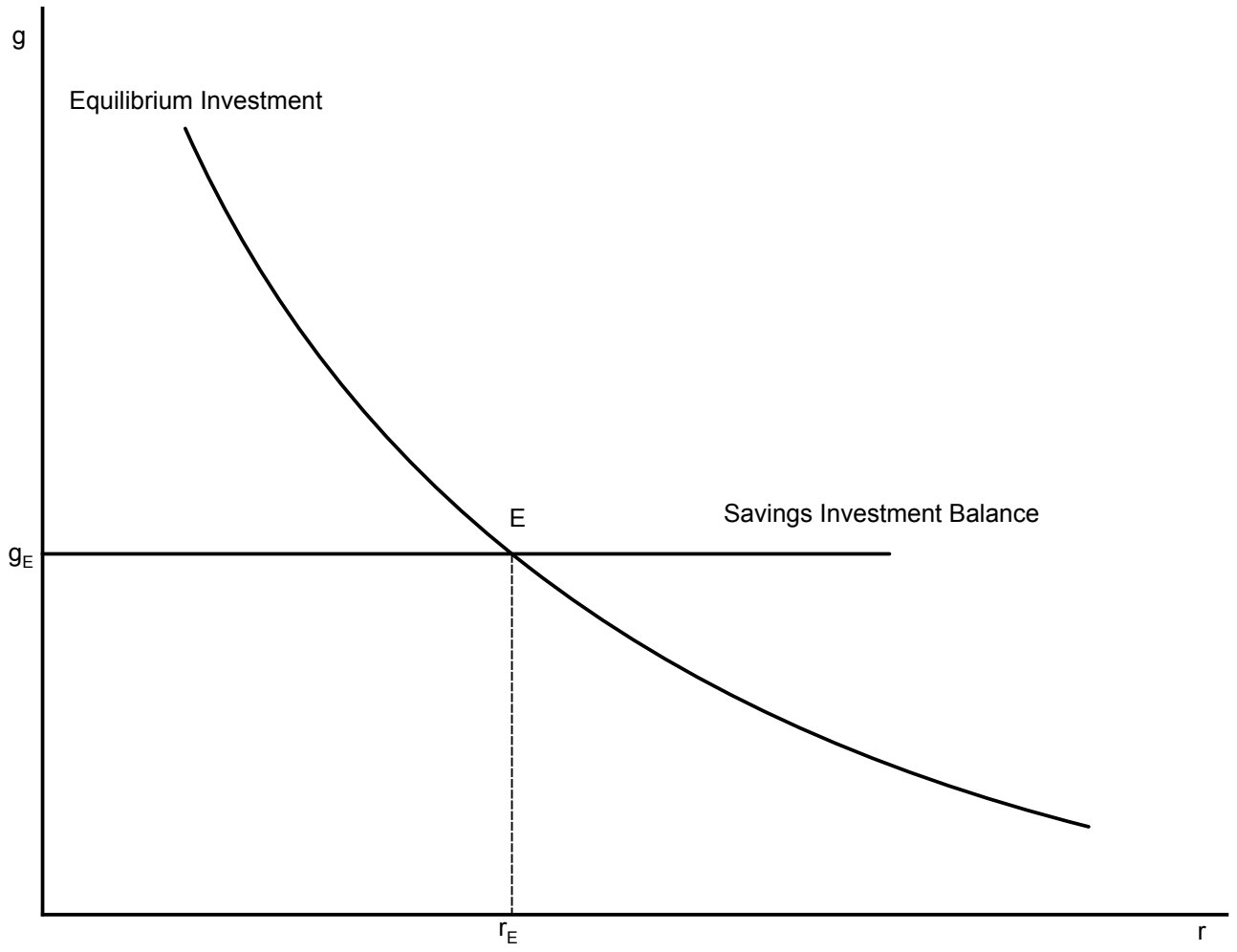


Figure 3
An Increase in the Savings Rate

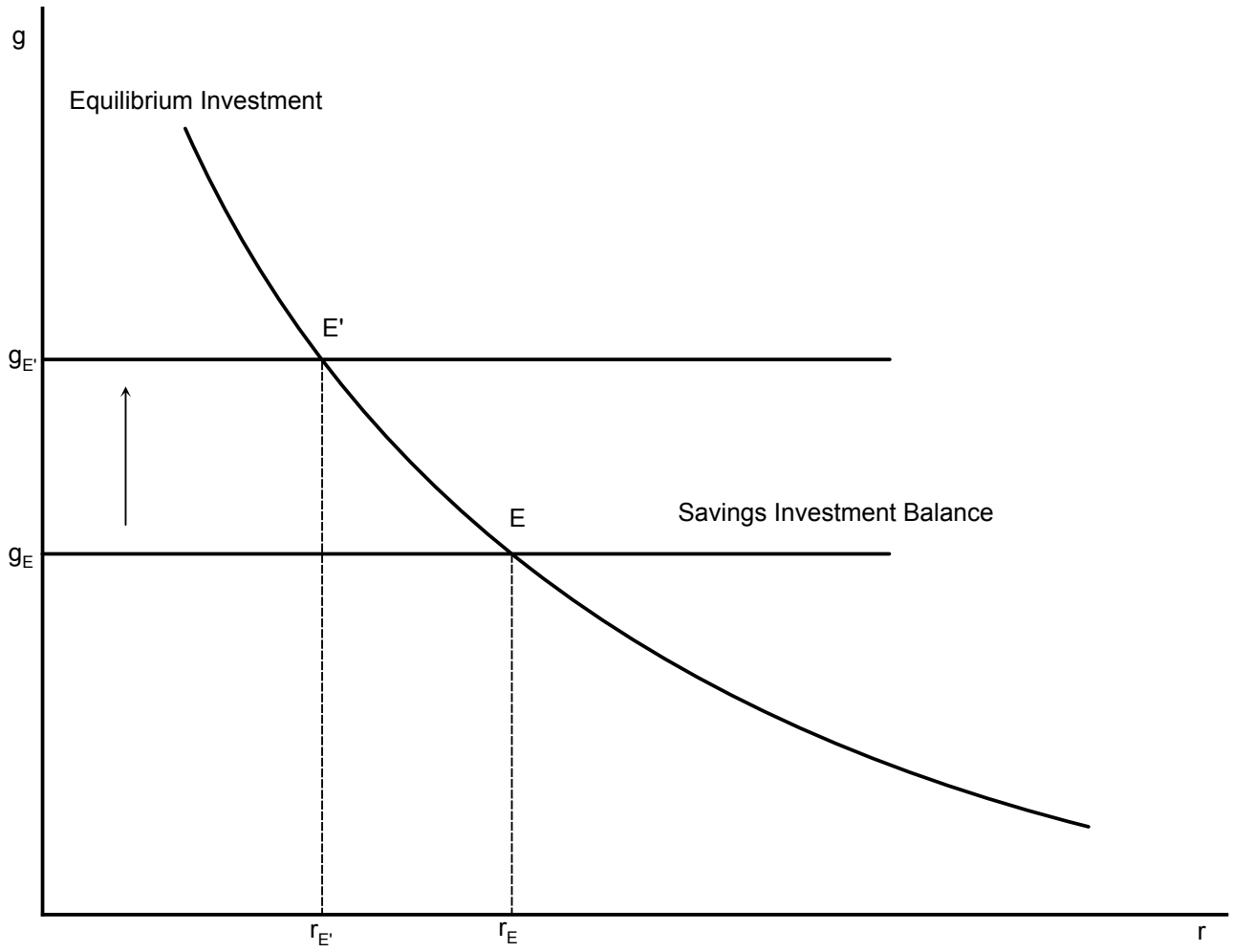


Figure 4
Equilibrium Investment and Savings in the Representative Household Model

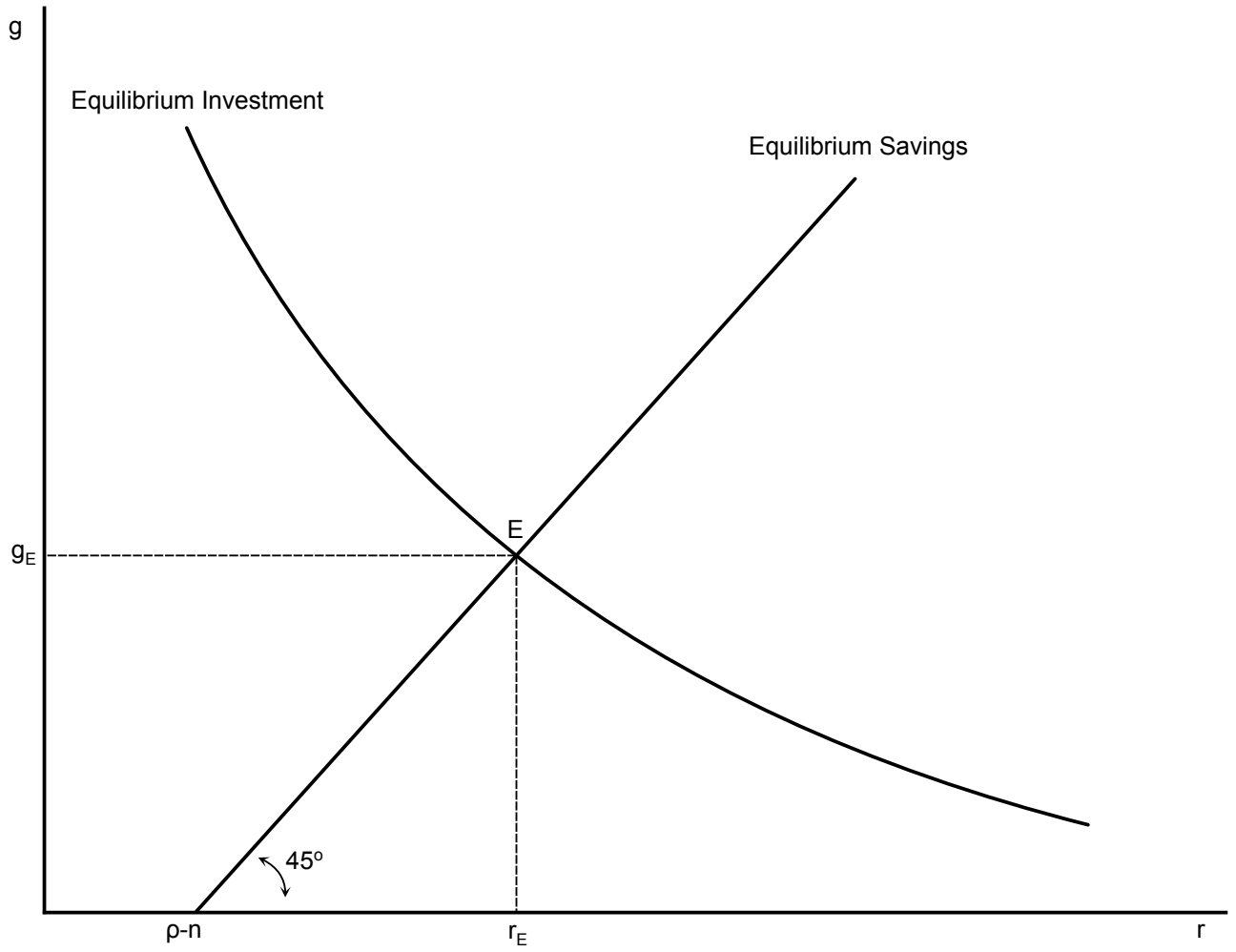


Figure 5
A Fall in the Pure Rate of Time Preference of Households

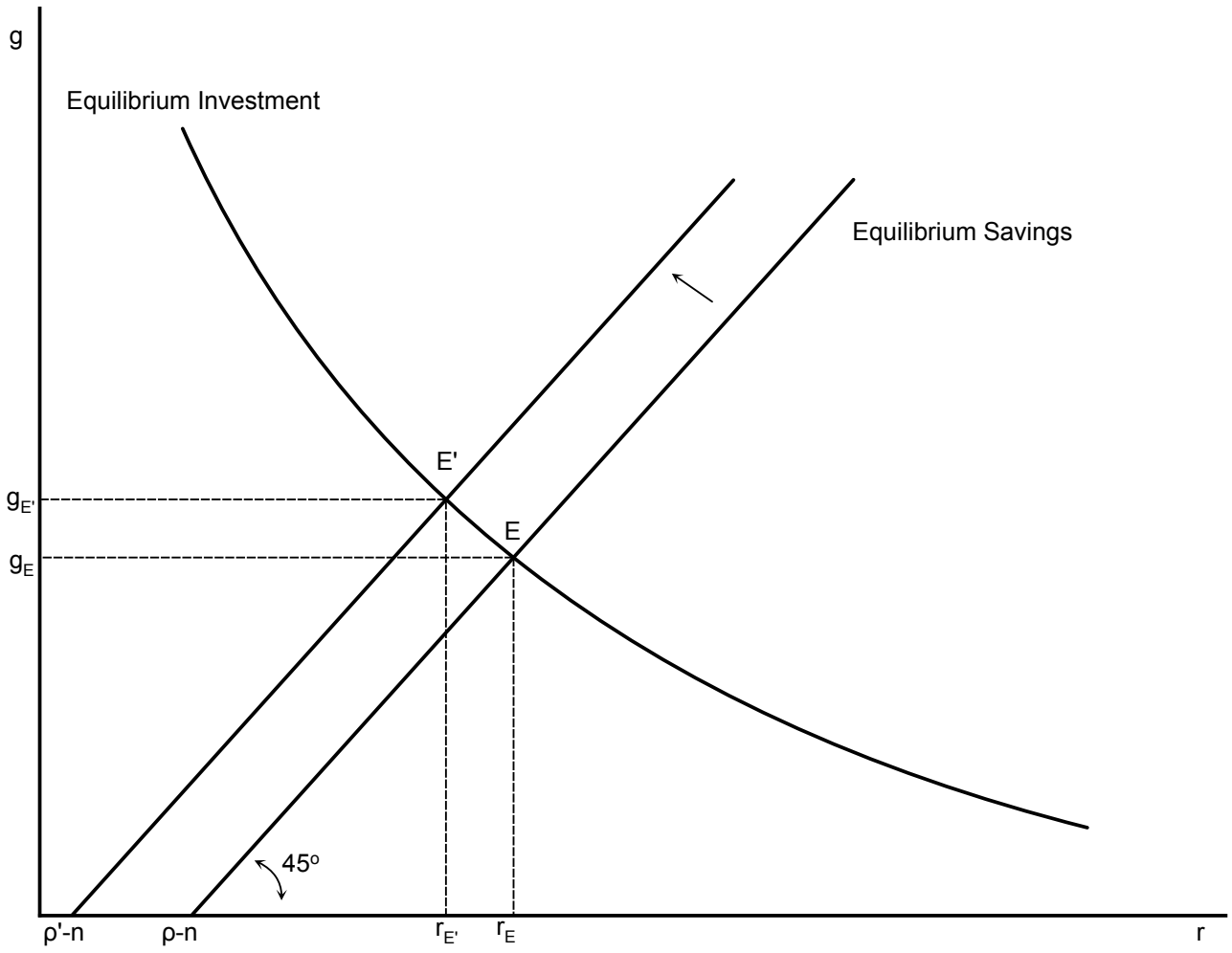


Figure 6
A Rise in the Marginal Productivity of Capital

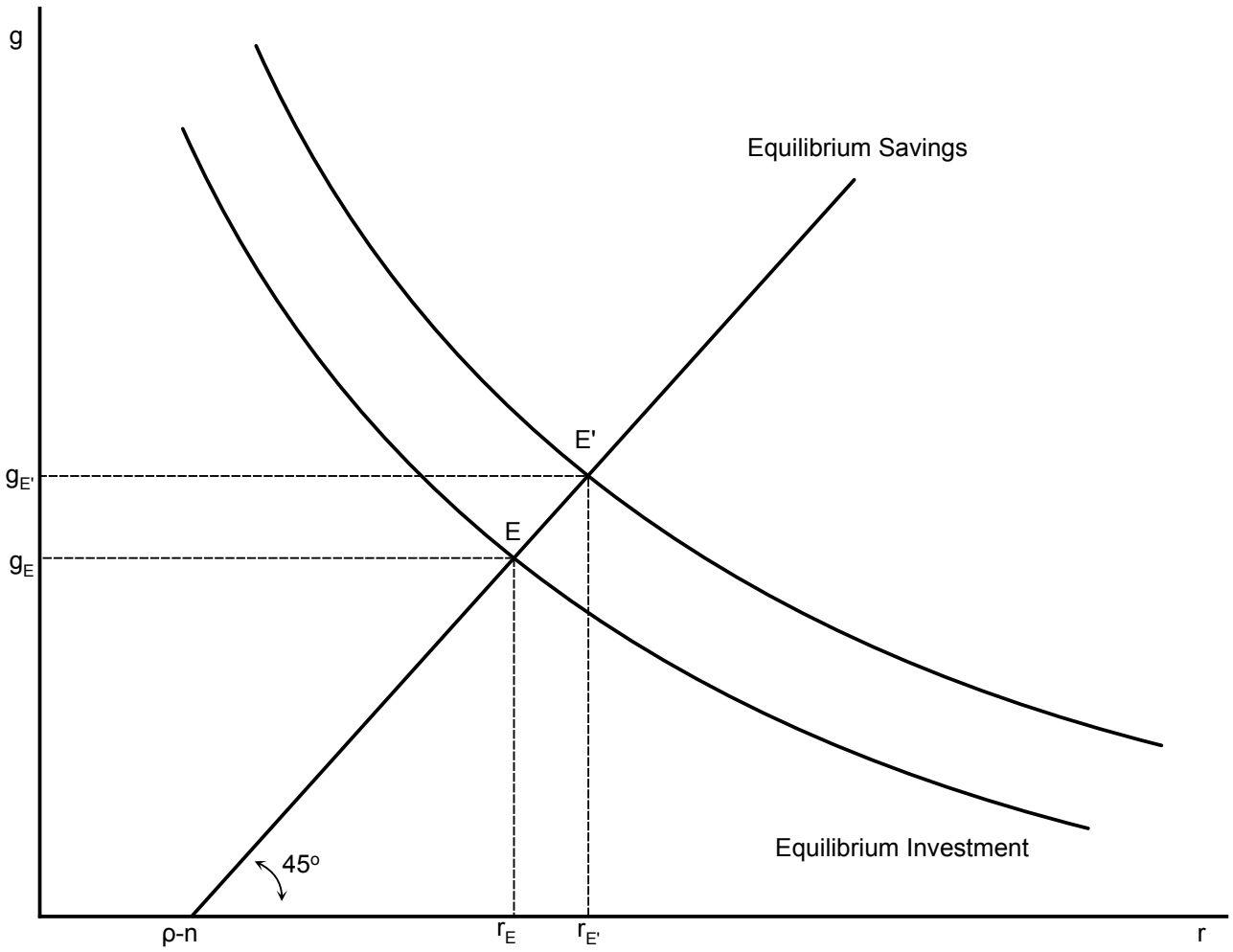
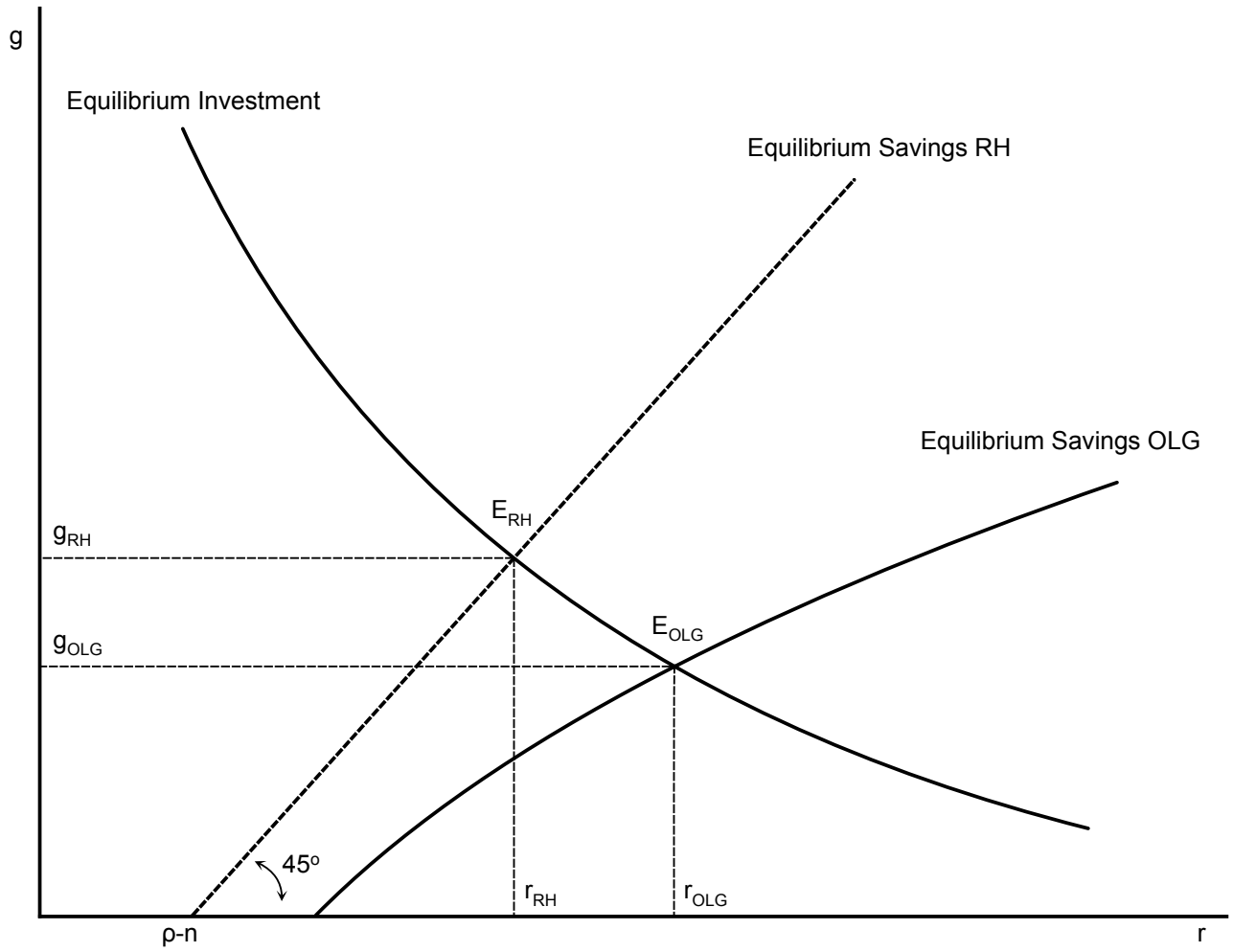


Figure 7
Equilibrium Investment and Savings in the Overlapping Generations Model



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