OPTIMAL PORTFOLIO MANAGEMENT

BY

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Abstract

The aim of this dissertation is to study the optimal portfolio management. Present the ways to select and evaluate an optimal portfolio of European stock indexes on the presence and the absence of a German Benchmark bond and compare it to the ASE index. Also present the calculation of optimal weights and compute the efficient frontier on both cases and the tangent portfolio, using the mean-variance theory of Markowitz that was the foundation of the modern portfolio theory.

First this dissertation introduces basic definitions of finance and all the mathematical background and equations to describe them. Then follows the presentation of modern portfolio theory and basic researches through the years until recently. In the second chapter we insert and analyze the notions of risk and ways to measure it as well as the immunization against it.

After we examine a more recent and complex method of optimizing a portfolio called dynamic stochastic programming and conclude that it is a computationally difficult method. Furthermore we show the methodology of the research by presenting the mean-variance method and all its calculations extensively. Finally in the last chapter we present and evaluate the results of the research and we used Microsoft Excel's Solver Add-Inn to compute all the optimization problems.

Keywords: Modern Portfolio Theory, Markowitz, Mean-Variance Theory, Tobin, Risk Measures, Performance, Efficient Frontier, Optimization.

Dedication

I would like to express my gratitude to my parents and my brother who supported and encouraged me during my studies. I would also like to thank my friends for all their help and support all this time. Finally I would like to thank my Prof. Elias Tzavalis for all his help during the drafting of this dissertation.

Chapter 1st

INTRODUCTION

1.1. How an investment works

In Economics investment means depriving an equal amount of resources from the current consumption in expect of growth in the future consumption. Investment is a financial flow by causing a financial outflow in the first period aiming in a range of cash inflows in subsequent periods, under uncertainty .The logic of the investment is to transfer part of our income in the future by reducing consumption in the present in order to improve our living standards, protect our capital from inflation and increase our safety against future economic adversities. Generally investment is a commitment, in a certain time period, of money or other assets in expect of profit in the future. The investor commits part of their capital expecting to gain profit by a certain investment in the future, by taking a certain level of risk to lose part of his invested capital or in a worst case scenario all of his invested capital. Investor's goal is to achieve the maximum return of an investment for a given level of risk that is willing to take. The whole meaning of an investment is to find the best balance between risk taking and return, given the resources, the goal, the preferences and the type of its investor.

Investment could be made either in capital goods[1], in securities[2], in currency, and derivatives such as put and call options. In modern financial theory investors are in the following basic categories. Retail investors or an individual is one category, they could be the consumers themselves and they take investing decisions for their own gain. In this category are also collectors of art or other items of value and angel investors, which are individuals who provide capital in a start-up business in exchange for convertible debt or ownership equity. Institutional investor is another category. In this category could be included banks, pension funds, insurance companies, hedge funds and mutual funds. The role of institutional investors, in opposition to retail investors is to act in behalf of others and according to their clients' preferences, they use their specialized knowledge and skills in financial management to advise them in what assets or securities to invest and when, so as to gain more profits for their clients.

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[1] Real estate, commodities etc.

[2] Bonds, stocks, certificate of deposits (CDs), treasury bills etc.

1.1.1. Types of institutional investors.

- Banks: Most of their investments concerning loans to entrepreneurs and consumers. To finance the investments use the savers' deposits and receive loans from the Central Bank or from interbank market. The banks set the interest rate of their loans higher than the interest rate they provide on the deposits and so they are hedging the risk of losing the capital they are lending to the risk of not fulfilling their obligations to their depositors and their lenders.
- Insurance companies: They are selling insurance contracts to a person or a firm who gets insured and pays an amount of money for insurance coverage, called premium. The insurance companies take the money and invest them mainly on bonds with the same maturity time as their obligations to their clients.
- Pension funds: Is a fund that provides retirement income. They administer the employer's and employee's contributions. Mainly they have large amount of money to invest and they are the major investor in listed and private companies, usually they invest their money in bonds and stocks with a predetermined guaranteed return.
- Hedge funds: is an investment vehicle that receives capital from a number of investors and invests in securities and other instruments.
- Mutual funds: is an investment fund that receives capital from many investors to purchase securities. Mutual funds in contrast to hedge funds are sold publicly.

The importance of Institutional investors in corporate governance has increased since the second half of the nineties as Huyghebaert and Van Hulle [2004] mention in their research. Different studies have investigated the importance of institutional investors in firms such as Gillian and Starks [2003], O'Neill and Swisher [2003] and they have shown that institutional investors affect the price of the company's share and liquidity and furthermore reduce information asymmetries between the firm and other investors. The amount of financial resources that are under control of institutional investors is constantly rising especially in the equity of listed firms in all OECD[3] countries as Huyghebaert and Van Hulle [2004] said in their research. The role of institutional investors in financial markets is rather important.

As far as risk is concerned there are three basic types of an investor:

- Risk averse: the investor who detests being under a great deal of risk taking and prefers the investment with lower risk.

[3] Is an international economic organization formed of 34 countries to trigger economic progress and world trade

- Risk seeking: the investor who moves toward risk and so prefers the investment with higher risk.
- Risk neutral: the investor who neither leans towards risk nor against it.

1.2. Risk vs. Return

Risk is a very important factor at investment selection. In finance, risk is defined as the variance of the results of an investment, as far as return is concerned, around its expected return. In a more basic way risk is defined as the chance of financial loss. Generally risk refers to the variability of returns of a given asset under uncertainty. As all managers know, when money is invested, there is a trade-off between risk and return and the greater the risks taken, the higher the return that could be accomplished. To be exact the trade-off is between risk and expected return not actual return.

The most well-known definition of risk is the one that Frank Knight [1921] wrote in his research in a period that was active in researches in the foundations of probability, he noted as risk the measurable uncertainty and the immeasurable one called it uncertainty. Risk according to Knight [1921] relates to objective probabilities and uncertainty relates to subjective probabilities. Other researchers more recently defined risk through their studies as Glyn A. Holton [2004], who concluded that operationally we can only define our perception of risk and it is without meaning to ask if a risk metric captures risk, it is better to ask if it is useful. Modigliani and Pogue [1974] also were involved with the meaning of risk in their study and noticed that how we treat risk is the key element in financial decision making. Furthermore Parker and Stewart [1974] explained in their research the relationship between risk and investment performance, as it seems defining and measuring risk is of high importance in portfolio management.

As it was mentioned earlier in the paper investors are categorized according to their aversion in risk. It is related to the reluctance of investors to accept an opportunity with uncertain return. Risk averse investors tend to avoid any kind of risk and prefer smaller and assured expected returns. They are only willing to take additional risk, when the expected return is particularly important. On the other hand risk lovers will always prefer taking higher risks even if there is an alternative investment with a slightly smaller expected return but with a lot smaller risk and finally there are risk neutrals that are indifferent to risk fluctuations according to expected return. Risk neutrals tend to believe that the higher return is closely intertwined to taking higher risks and in comparison to risk averse investors they take additional risk if they expect proportionally sufficient additional benefits.

1.2.1. Risk and its types.

A typical subject that concerns investors in portfolio selection is the various sources of risks which will be presented below as they are categorized in modern portfolio theory:

- Interest rate risk: It is the risk rising of fluctuations in interest rates that will adversely affect the value of an investment in case that all the other factors affecting the prices of securities remain constant. Mostly investments lose value when the interest rate rises and increase in value when it falls.
- Market price risk: The risk that the market price of assets will change with time. Depending on the type of asset we distinguish between stock market price risk and fixed income market price risk.
- Stock market price risk: The risk that the price of a stock will change with time due to adverse movements of the stock market of the stock market as reflected in market index changes.
- Fixed income market price risk: The risk that the price of a fixed-income security will change with time due to adverse movements of the fixed income market as reflected in market index changes.
- Shape risk: It is the risk that the price of a security will change with time due to changes in the shape of term structure of interest rates.
- Volatility risk: It is the risk that the price of an asset will change with time due to changes in volatility.
- Credit risk: It is the risk of an unkept payment promise due to default of a counter-party, issuer or borrower, or due to adverse price movements of an asset caused by an upgrading or downgrading of the credit quality of an obligor that brings into question their ability to make future payments.
- Currency risk: It is the risk that the price of a security will change with time due to changes in the exchange rates between different currencies.
- Liquidity risk: The risk arising when a firm is unable to raise cash to fund its business activities (funding liquidity) or cannot execute a transaction at the current market prices due to a lack of appetite by other market players. Liquidity is affected by the size and depth of the market in which an investment is traded.
- Inflation risk: It is known as purchasing power risk and has to do with uncertainty about inflation, it affects all securities. Changes in price levels caused by inflation or deflation in economy will affect the cash flows and value of an investment. The actual yield obtained after deflating contains risk even if it is a no risk security, due to the fact that the rise in inflation has raised the interest rates too.
- Business risk: It is the risk arising due to volatility of volumes, margin or costs when engaging in the firm’s business.
- Operational risk: This is the risk of direct or indirect losses resulting from inadequate or failed internal processes, people and systems and from external events.

- Sector risk: The risk of price movements affecting a group of securities that share some common characteristics.
- Residual risk: This is the risk of price movements due to firm-specific effects and unrelated to the systematic influences given in our list of risks.
- Actuarial risk: This is the risk associated with the liability side of the balance sheet of insurance firms and it exists due to changes in mortality, causality or liability exposures.
- Systemic risk: The risk of wide-spread collapse or dis-functioning of the financial markets through multiple defaults, widespread disappearance of liquidity, domino effects etc.
- Tax risk: If unfavorable changes in tax laws occur the investments that are sensitive to tax law changes are riskier.

Furthermore we need to introduce to notions regarding the risk of the portfolio. The total risk of a portfolio consists from two parts, the systematic and the non-systematic risk. The systematic risk is the risk that can be eliminated or at least a large part of it can be minimized if we have a large portfolio with several securities through diversification. The other part of the total risk of a portfolio is the non-systematic or else idiosyncratic or specific risk and is the part of the risk that cannot be diversified away.

1.2.2. Return of a portfolio.

Another crucial term in portfolio management as mentioned earlier that is deeply connected with risk as far as investment performance is concerned is return. The notion of return is defined as the percentage change of the investment value during a given time period, it is the driving force in the investing process and the reward of investors for taking the risk.

The realized return is the actual return on an investment held in a passed period, so the prices of securities were known. The return R_T of a security at time t is equal to the ratio of change in price, in the time period we are studying as to its initial value.

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1.1}$$

Where R_t is the return of the security from time period t-1 until time period t, P_t is the final price of the security at time period t and P_{t-1} is the initial value of the security at time period t-1.

When investors must make up their minds whether to invest in a security or not, they have to study the relationship between risk and return and also examine their alternatives. A major problem is that they cannot know the future price of the security and therefore neither the future performance, in which they are interested at. Not knowing the future price of a security is how we concluded to the outcome of estimating the future value, by using information of the historical behavior of the security's price with the help of the method of forecasting. If we want to make an estimation we use a predictive model and during this process enters the risk, in this case risk is the probability of security's future price deviates from the predicted, by the model, price. This is the point where we are going to insert the notion of expected return. Expected return is the return that investors predict they will derive in the future from an investment. To calculate the expected return from a sample of n past periods we use the mean, if the returns considered equally likely to happen the expected return is given by the following type:

$$E(R_t) = \frac{1}{n} \sum_{i=1}^{n} R_{it} \qquad (1.2)$$

In case where returns of a security are not considered equally likely to happen and each return is weighted by the corresponding probability P_i to occur, the expected return is the weighted average of historical returns R_{it} given by the following type:

$$E(R_t) = \sum_{i=1}^{n} P_i R_{it} \qquad (1.3)$$

Through the explanation of risk and return it was mentioned the term of the security, which is an investment traded product issued by a government a company or other organizations. Security is a financial instrument that is evidence of debt or ownership, which means they are categorized into debt securities and equities, issued by a company or entity, who wishes to raise funds from retail investors. Some very widely known securities are: bonds, shares, treasury notes, option contracts, future contracts and other products that can be traded on a financial market. Each security as a unit has an expected return and contains risks so it will be examined by the mean and the standard deviation of its performance.

A portfolio is a set of securities that a company or an entity has in his possession. Holding a portfolio of securities instead of separate securities the investor has the ability to mitigate the impact of a negative change in the price of a security in the value of the portfolio. That happens because the negative changes in the price of one security could be neutralized by the positive changes in the price of others, due to the benefits of securities' diversification in a portfolio. The impact of extreme fluctuations in the market could be insignificant in the value of a portfolio with a large number of securities and a significant diversification of securities, that's why the investment in a portfolio involves lower risk of capital loss than in a separate security. The benefits of portfolio diversification are very important.

Portfolio as a set of securities has risk and expected return. Expected return of portfolio is given by the following type:

$$E(R_p) = \sum_{i=1}^{n} w_i E(R_i) \tag{1.4}$$

Where $E(R_p)$ is the expected return of portfolio p, $E(R_i)$ is the expected return of the i-th security, w_i is the weight of the i-th security of the portfolio, n is the number of securities in the portfolio.

The calculation of portfolio risk σ_p is much more complicated and it will be explained in the second chapter of the study. Risk and return are important factors in portfolio selection. The selection of a portfolio relies on the decision of which securities will participate in it and in what proportion each so as to have a portfolio with the best possible expected return and risk. That subject concerned many great minds through the years and we will talk about them later in the study.

1.3. Portfolio strategies and types

The objective of portfolio management is the application of diversification and to create portfolios with maximum expected return and minimum risk under uncertainty. Efficient portfolio is the one that for a given level of risk has the maximum possible performance and vice versa for a given level of return presents the minimum risk. Portfolio management developed in the late 1950s and it is an extension of financial theory. The portfolio theory is based on the research of H. Markowitz [1952] about the selection of an optimal portfolio. By the term portfolio management we refer to all the necessary actions that an investor has to fulfill to create a portfolio to secure the invested capital. Portfolio management is the process of choosing the appropriate in a portfolio regarding the investors' preferences, its monitoring and performance measurement. Portfolio management has three basic steps:

- ❖ Securities analysis: The examination which of the available securities are expected to have the highest performance.
- ❖ Portfolio analysis: Prediction of portfolio's return and risk, regarding the securities of which it consists.
- ❖ Portfolio selection: From the portfolios that minimize the risk regarding to return is selected the one that fits to the investors characteristics, for example how much money he wants to invest and for how long.

Regarding the portfolio strategies there are two basic forms:

- Passive Portfolio Strategy: A strategy that barely involves forecasting data, on the contrary, it is based on diversification to hedge the performance of a market index. A passive strategy believes that all the available information is reflected in the price of securities. Passive strategy is widespread in the stock market, but it is more often used in other types of investment such as bonds and hedge funds.
- Active Portfolio Strategy: This is a strategy that uses available information and forecasting techniques to achieve better return than a portfolio that is simply broadly diversified. It refers to a portfolio management strategy, when a manager makes specific investments in order to outperform an investors' benchmark index. According to the objectives of the investment portfolio that will be created, the active strategy helps to create a smaller risk from the benchmark, seeks to exploit any poorly aimed pricing of securities and simultaneously to be sold securities that the managers considers them to be overvalued.

Moreover there are three portfolio types:

- Patient Portfolio: This type of portfolio invests in well-known shares, which are highly capitalized. Most of them pay dividends and they are to be bought and maintained for long time periods. The majority of those shares in this portfolio represent formal development companies that are expected to generate higher profits on a constant basis regardless of the economic conditions.
- Aggressive Portfolio: This portfolio type invests in "expensive" stocks, in terms of measurement such as the price to earnings ratio, which offer greater surplus value but also involve higher risks. This portfolio collects shares of rapidly growing companies of all sizes that are expected in the next few years to experience a rapid growth in their annual profits. This portfolio is most likely to show large cycles of changes over time and it is easy to distinguish "losers" from "winners", because its shares belong to the not so well established ones.
- Conservative Portfolio: In this portfolio shares are chosen based on their performance, the earnings growth and a history of a stable dividend.

1.4. Modern Portfolio Theory: The mean variance theory, CAPM and APT theory

The theory of the portfolio is an investment approach developed by economist Harry Markowitz [1952] at the University of Chicago. Harry Markowitz shared the Nobel Prize in 1990 with William F. Sharpe and Merton H. Miller, U.S. economists, for their work in the theory of financial economics. The portfolio theory allows the investors to estimate both expected risks and expected returns as measured statistically for their investment portfolios. Markowitz [1952] described the way in which the data are combined in efficiently diversified portfolios and pointed out that the risk of portfolio could be limited and the expected rate of return could be improved if investments with unequal prices' variations were combined. Markowitz model despite its weaknesses has been the keystone to Modern Portfolio Theory. The publication of Markowitz in the newspaper "Journal of Finance" in 1952 brought undeniable crucial changes, created a new era and a new way of thinking. Furthermore Markowitz [1959] reached some conclusions in his book, entitled "Portfolio Selection" in 1959.

1.4.1. Mean-Variance Theory

Markowitz presented a model of constructing efficient portfolios and the baseline was the selection of an optimal portfolio of shares or other investments that involves risk, which offers the best possible risk-return relationship. The optimal portfolio is the one that maximizes the investor's expected utility in a single period of time. The model applies under some assumptions:

- The investor has a specific investment period.
- An investment portfolio that consists by individual stocks is described completely by the expected return and the variance of its performances.
- Investor perceives each and every security by the security's probability distribution of expected returns. The expected distribution's price defines the security's expected return and the variance of the distribution as a risk measurement of the security. Also the investor estimates the covariance between the securities' expected returns that are included in the portfolio.
- The investor's goal every time period is the maximization of his utility function. The utility function is based on the price and variability of portfolio's expected return and that is why the investor prefers the best possible return for a certain risk level or the minimum risk for a certain level of return.
- Investors are risk averse and rational.
- The investor prefers consumption growth.
- The investor's utility function is concave and ascending due to the aversion to risk and the consumption growth preference.

Markowitz models the portfolio selection process by separating it into two main steps. The first step starts with the experience and observation and ends with the investor's estimations on the available securities future performances. After the first step comes the second, that deals with the selection of the portfolio and that was the main theme of Markowitz publication, the selection of the optimal portfolio given the investor's requirements. Markowitz defines risk as the volatility of securities' prices over the years that cannot be eliminated through diversification, disproved the theory based on the law of large numbers and noted that the expected return is not the only criterion for the selection of the proper investment and that is extremely unlikely to occur a portfolio which at the same time maximizes the expected return and minimizes the risk. The main point of the research is to find combinations for which, for a given level of return there is no other combination with lower risk, or vice versa, for a given risk level, other combination with higher return. Moreover Markowitz rejects the theory that the objective of the investor is only profit maximization.

Markowitz defines expected return of a portfolio E and variance V by the following types:

$$E = \sum_{i=1}^{n} X_i \mu_i \quad (1.5)$$

$$V = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} X_i X_j \quad (1.6)$$

By μ_i symbolizes the expected return of each share and by σ_i the covariance of the returns of i and j securities. The covariance σ_i is given by the equation:

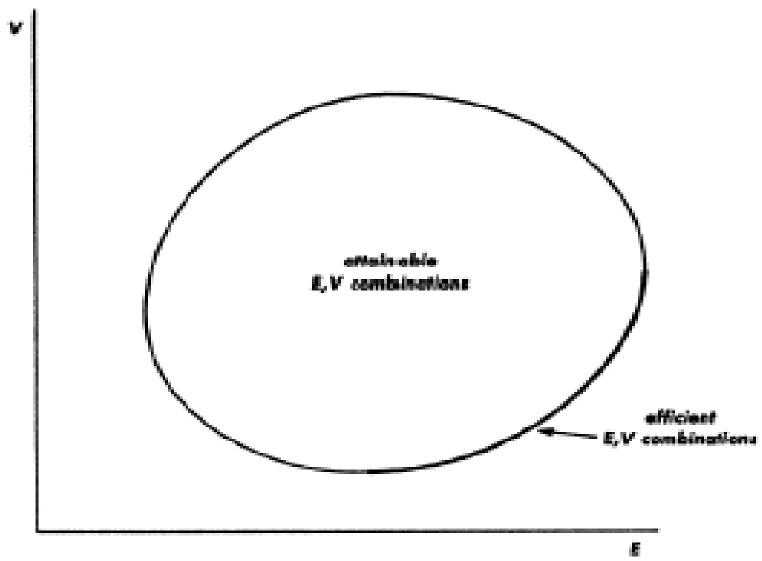
$$\sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \quad (1.7)$$

Where σ_i and σ_j are the standard deviations of the securities and ρ_{ij} is the correlation coefficient. The correlation coefficient range is [-1.1]. When the returns of the two securities are linearly uncorrelated then the correlation coefficient equals zero. The maximum value -1 suggests that the returns of the two securities are linearly correlated and move in the same direction, while the minimum value -1 means that they move in the opposite direction. Also the participation percentages (weights) of each investment in the portfolio are not random and they are determined by the investor and symbolized by x_i.The entire amount is invested by the assumption and so applies the following:

$$\sum_{i=1}^{n} x_i = 1 \quad (1.8)$$

According to the E-V rule the investor has the option to choose one of those combinations, which are within the circular area of the Figure 1.1 and called attainable E-V combinations. The efficient E-V combinations are represented by the intense black border, they are the combinations that for given E have minimum V and for given V have maximum E. Markowitz did not presented the techniques of computing optimal portfolios and their efficient E-V combinations given μ_i and σ_{ij} , he presented the geometrical specification of effective surfaces in the case where the number of securities N is small. The first specification of Markowitz [1952] concerns a portfolio consisting of three securities and he solved it using two-dimension geometry. The attainable set of portfolios are represented by the triangle abc, as we can see in Figure 1.2. The lines, called isomean lines, as shown in Figure 1.2, which represent the set of portfolios with equal efficiency are dotted and parallel and the ellipses, called isovariance curves, which contain the portfolios with equal variance are concentric in the center X. X is the point with the smallest standard deviation and is the center of the system. The tangent point of an isomean line with an isovariance curve is the one with the minimum risk. If we unite with a curve all the minimum risk points, the curve I created turn out to be a straight line called critical line. The efficient set of portfolios consists of two segments with intense black color as shown in Figure 1.2. The first segment is part of the critical line which starts from X and extends until the intersection with the straight line ab. The second segment starts from the intersection and continues until point b, the point with the maximum E. In case X is outside the triangle abc, as shown in Figure 1.3, the analysis is the same. Variance increases as we move away from X. The critical line 'cuts' the feasible region and the efficient line, the one with the intense black color, starts from the line ab and moves towards b until it intersects with the critical line and then they continue together until the limit of abc and the again changes direction and ends up to b.

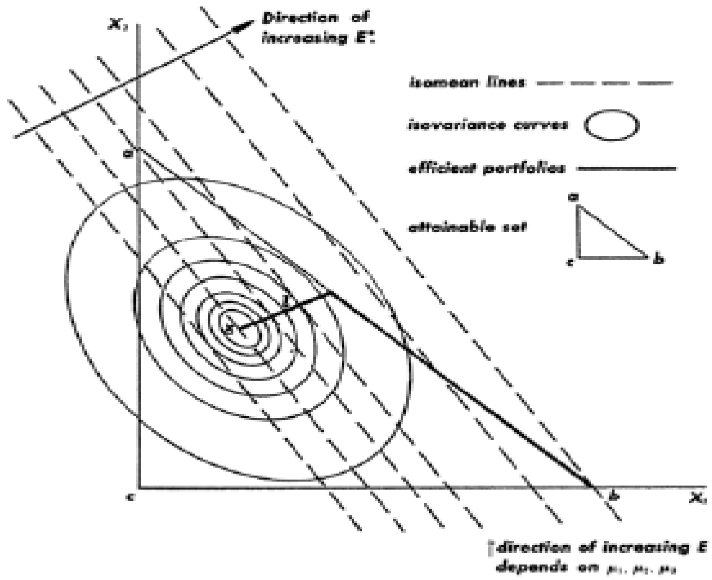
Figure 1.1: Efficient E-V combinations of the investor.



Source: From H. Markowitz essay "Portfolio Selection" published in the Journal of Finance, 1952

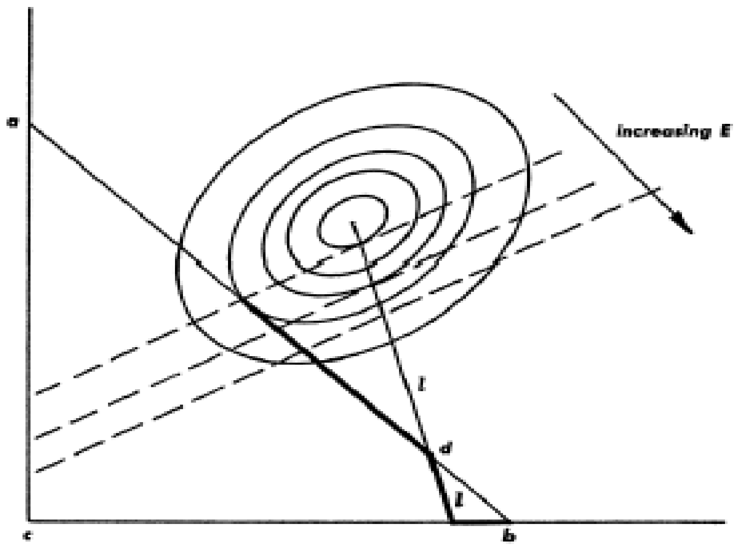
Markowitz [1952] concluded to the parabolic shape shown in Figure 1.4. That figure shows a convex curve which later in the modern portfolio theory proved to be a concave curve. Markowitz [1952] believes that diversification makes sense and brings the desired results only when done correctly and the E-V rule can be used either for theoretical analysis or for portfolio selection in stock market. Finally he believes that the key assumption to achieve the mean-variance analysis is the correct calculation of the expected return and risk, with correct historical data for securities.

Figure 1.2: Graphic portfolio solution with three equations and X inside the triangle abc.



Source: From H. Markowitz essay "Portfolio Selection" published in the Journal of Finance, 1952.

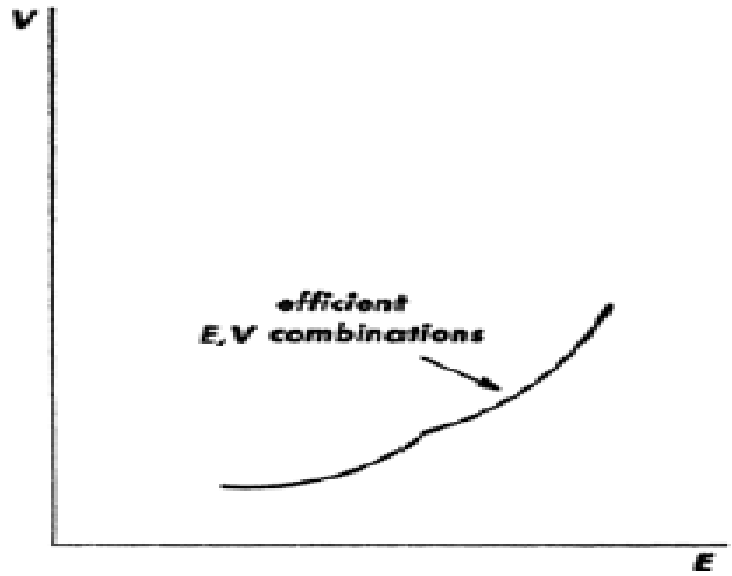
Figure 1.3: Graphical portfolio solution with three equations and X outside the triangle abc.



Source: From H. Markowitz essay "Portfolio Selection" published in the Journal of Finance, 1952.

Markowitz publication in 1952 made it a crucial year for the development of modern portfolio theory. In his book in 1959 based his theory on a more mathematical way and among other things analyzed the process of calculating the efficient frontier. Before Markowitz the approach was relatively simple, the investor chose each security independently from the others, based on the return and the risk. After Markowitz came Tobin in 1958 and introduced to the initial theory the risk-free asset. If we have a portfolio that is efficient and combine it with a risk free asset we can achieve greater performance with the same risk. In 1959 followed the publication of Markowitz book and the extension of his theory by Sharpe in 1964, who was one of the founders of the capital asset pricing model. The CAPM introduced the notions of systematic and non systematic risk. Other models followed such as the arbitrage pricing theory-APT by Stephen Ross [1976] and several studies following the direction of Markowitz model or different approaches as we will see later in the thesis.

Figure 1.4: Efficient E, V combinations.



Source: From H. Markowitz essay "Portfolio Selection" published in the Journal of Finance, 1952.

1.4.2. Tobin's risk-free asset

The economist James Tobin published his article "Liquidity Preference as Behavior towards Risk" in 1958 in the journal Review of Economic Studies and based on the mean-variance model developed a portfolio selection model with the main difference being the introduction of the risk-free asset. Tobin analyzes firstly in his article the reasons why investors maintain liquidity and to what percentage of their capital. Deposits and bonds are forms of investment in fixed yield, which are relatively small and risk free. The investor can avoid some of the risk by holding a portion of his funds in liquid form, which contains risk and the rest to invest in a portfolio of securities that have higher returns and risk.

The investor takes decisions and makes investment choices guided by his attitude towards risk and his utility function. Tobin [1958] added in his research the concept of leverage in the Markowitz portfolio theory, incorporating in the analysis the risk-free asset. Combining the risk-free asset security with a portfolio that belongs to the efficient frontier is possible to construct portfolios with even better return-risk combinations, this is possible through the asset or liability borrowing in the rate of the risk-free security. Tobin's separation theorem is one of the most important contributions to the economic area. Tobin separates the portfolio construction process in two stages. Initially investors should choose the securities that are the dangerous part of the portfolio and create an effective portfolio. Next you have to combine it with the risk-free security by selecting the degree of leverage they use, depending on the preferences regarding risk and form the final portfolio containing a safe part and one that is risky. According to the separation theorem these two processes are completely independent of each other, since neither of them affects the other.

1.4.3. The Capital asset pricing model.

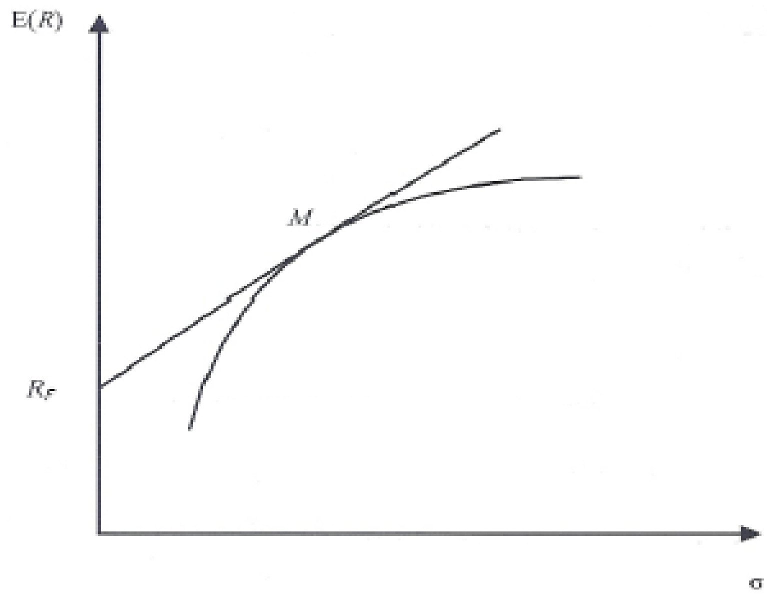
The CAPM, capital asset pricing model, developed simultaneously and independently by William Sharpe [1964], John Lintner [1965] and Jan Mossin [1966] and is the most common equilibrium model. In equilibrium models is reflected the way investors behave as a whole and not individually, as in the mean-variance model. Through equilibrium models can be determined a relative measure and the risk-return relationship of a security. The capital asset pricing model is based on the mean-variance model and consists by the following assumptions:

- There are no transaction costs.
- All assets are infinitely divisible.
- All assets are traded.
- There is no concept of personal tax income, so the investor is indifferent on the form of performance of an investment.
- No investor by himself can affect the price of a stock by selling or purchasing it (perfect competition hypothesis).
- All investors decide based only on the expected return and standard deviation of a portfolio.
- Short selling is allowed to an unlimited extent.

- All investor can lend or borrow at the rate of the risk free asset.
- All investors take into account the expected value and the variance of returns of the securities, based on a specific time horizon.
- All investors have exactly the same expectations for all entry data that are required to support a decision in portfolio selection.

The CAPM model defines the overall risk of an investment as the sum of diversifiable and non-diversifiable risk. The diversifiable or non systematic risk refers to the variation of returns of each security, regardless of overall market movements and can be reduced or eliminated through diversification. The non-diversifiable or systematic risk caused by macroeconomic factors that affect the returns of all risky investment and is inevitable, since any option of diversified portfolios involves systematic risk. According to the CAPM assumptions is logical for all investors to choose as risky portfolio the market portfolio. The market portfolio is the optimal risky securities portfolio, because it is perfectly diversified, since it consists of all market securities that each one has a participation percentage proportionate to its market value. The market portfolio belongs to the efficient frontier by Markowitz and its total risk equals to systematic risk. The market portfolio is a theoretical portfolio and for the measurement of risk and return are used stock market indexes, while it can be approached by a well diversified portfolio consisting solely of common stocks. Every investor places a percentage of the money in the market portfolio and the rest in the risk-free asset. All investors will have in their possession only combinations of two portfolios, the market portfolio and the risk-free asset.

Figure 1.5: The Capital market line



The straight line shown in the Figure 1.5 is called CML, capital market line, in which are all the efficient portfolios which will be chosen by all the investors under the assumption of the CAPM. The inefficient portfolios are below the capital market line. The equation of capital market line is the following equation of the efficient frontier:

$$E(R_e) = R_f + \frac{E(R_M)-R_f}{\sigma_M}\sigma_e \qquad (1.9)$$

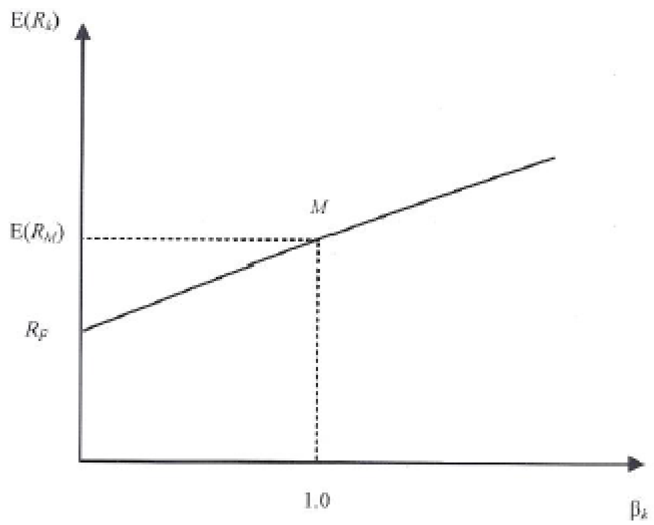
Where $E(R_e)$ is the expected return of an efficient portfolio e that belongs in the capital market line, R_f is the return of a risk-free asset F, $E(R_M)$ is the expected return of the market M, σ_E is the standard deviation of efficient portfolio e of the capital market line and σ_M is the standard deviation of the market M.

The first term of the product of the equation is the market price of risk, which is the additional return that the investor requires to win when there is a unit increase in the level of risk in an efficient portfolio. This equation shows that the expected return of an efficient portfolio is the sum of the return of the risk-free asset and the product of the market price of risk on the portfolio risk. To calculate the returns of the inefficient portfolios and the individual securities, which do not belong to the capital market line, the capital asset pricing model is using the following equation which responds to the line of the Figure 1.6 and is called SML, securities market line.

$$E(R_i) = R_f + \beta_i\big(E(R_M) - R_f\big) \qquad (1.10)$$

Where $E(R_i)$ is the expected return of a security or a portfolio i, R_f is the return of risk-free asset F, $E(R_M)$ is the expected return of the market M, β_i is the beta coefficient of the security ,where $\beta_i = \frac{\sigma_{iM}}{\sigma^2{}_M}$.

Figure 1.6: The security market line.



The beta coefficient is a measure of systematic risk of an investment equal to the ratio of the covariance of the returns of an investment to the variance of the market portfolio. The beta coefficient measures the sensitivity of the return on capital elements to the change of the overall market return, the non-diversifiable risk. When an investment has beta equal to one if the market moves up or down by a percentage the investment will also move up or down by the same percentage. The market portfolio as shown in the Figure 1.6 has beta equal to one while the securities with beta greater than a unit are called aggressive because they are riskier than the market portfolio. The securities with beta less than a unit are called defensive and show lower risk than average. As a conclusion CAPM was based under some strict assumptions and it provide a good estimate of the risk and the return of an investment or portfolio but the reliability and accuracy of the model deviates from the actual data.

1.4.4. The Arbitrage pricing theory.

The CAPM is not the only model of balance and valuation of capital assets when the market is in equilibrium. The APT, arbitrage pricing theory was formulated by Stephen Ross in 1976 and it is a generalization of CAPM. Ross [1978] wrote many articles on APT and also Ross [1977]. While the CAPM identifies as the only source of risk the systematic and non –systematic risk, expressed through the beta, Ross found that the risk-return relationship is much more complicated due to the existence of several sources of risk and set the return on a stock as a result of several economic factors. Through the arbitrage pricing theory is achieved the transition between from multi-index model in an equilibrium model. The APT is based on the mean-variance model and the market portfolio as the CAPM but is also based on the law of one price, according to which two identical assets cannot be sold at different prices. The law in other words says that the same title cannot be traded in markets at different prices. The returns of the securities are connected linearly with a set of indexes, each of which represents a factor that affects the performance of the security. Investors have same expectations for the sensitivity of the securities related to these factors and the buy and sell securities that are affected equally by the same factors and have equal expected returns. The process of buying and selling is the process of arbitrage and determines the equilibrium prices of the securities in the market. The arbitrage pricing theory has an advantage next to the CAPM, it has fewer restrictive assumptions. The assumptions of CAPM that do not apply on APT are: The investment time horizon of one year, the absence of taxation, active and passive lending at the rate of the risk-free security, investors select portfolios based on the expected return and variance. The APT as a factor model defines the behavior of the prices of securities by recognizing the risk factors of the economy that influence the realized and expected return of securities. The risk factors represent broad economic forces. According to the arbitrage pricing theory the investors believe that the realized returns of securities are generated randomly according to an n-factor model with the following equation:

$$R_i = E(R_i) + \beta_{i1} f_1 + \cdots + \beta_{i\,n} f_n + e_i \qquad (1.11)$$

Where R_i is the realized return of a security i in any time period t, $E(R_i)$ is the expected return of security i, f_n is the deviation of the n-th systematic factor from its expected return, $f_n = F_n - E(F_n)$, β_{in} is the sensitivity of security i according to its n-th factor F and e_i is the random error term unique for its security.

We consider that the error terms e_i of the securities are uncorrelated because the covariances of all securities are results of the factors. The expected value of its factor F is equal to zero, therefore the actual performance of a security i at any time will be equal to its expected return, when the risk factors are the expected levels (f_i =0) and the error term e_i is equal to zero. The equation 1.12 is based on the equation 1.11 and it is the equilibrium model.

$$E(R_i) = a_0 + b_{i\,1}F_1 + \ldots + b_{in}F_n \qquad (1.12)$$

Where $E(R_i)$ is the expected return of security i, a_0 is the expected return of a security with zero systematic risk, F_n is the risk premium of the n-th factor F, $F_n = E(F_1) - a_0$ and b_{in} is the sensitivity of the security i according to the risk premium of the n-th factor F.

The problem with the APT is that risk factors are not certain either qualitatively or quantitavely in the first period and it does not specify the size and sign of the factors and the risk factors should be evaluated empirically. A very important research for the application of the APT in practice was made by R.Roll and S.Ross [1980].

1.5. Modern Portfolio Theory: Other researches until recently

We will present in brief some other important researches through the years until recently.

1.5.1. Roll, Ross, Chen factors.

In 1986 Richard Roll, Stephen Ross and Nai-Fu Chen identified the following macroeconomic factors to explain the performance of securities:

- Unexpected change in inflation
- Unexpected change in interest rates
- Unexpected change in the Gross National Product.

- Unexpected shifts in production curve.
- Unpredictable change in the investors' confidence because of changes in default premium.

1.5.2. Fama-French model

In 1992 Gene Fama and Ken French developed the model of three factors called Fama-French three factor model to describe the behavior of the market and of the portfolio normal returns. The CAPM used only one factor, the beta, to compare the excess returns of a market began observing two classes of shares that tend to be better than the overall market. The two categories of shares are the small caps and the stock with a high book-to-market ratio. By adding these two factors to the CAPM they concluded to the following relationship for the estimation of the return of a portfolio.

$$E(R_t) = R_f + (H_{beta} - L)\beta_{beta} + \left(S_{cap} - L_{cap}\right)\beta_{size} + (L_{PBV} - H_{PBV})\beta_{PBV} + e \quad (1.13)$$

Where $H_{beta} - L$ is the volatility in the market return, $S_{cap} - L_{cap}$ is the size of the asset relatively to the capitalization of the market, $L_{PBV} - H_{PBV}$ is in terms of meditative value, e is the positive or negative error term.

1.5.3. Carhart's model

Later in 1997 Carhart M. added a forth factor to improve the Fama-French model, regarding to the short-term prediction of return. This factor is the momentum which is the continuation of a trend and concluded to the following condition:

$$E(R_t) = R_f + (H_{beta} - L)\beta_{beta} + \left(S_{cap} - L_{cap}\right)\beta_{size} + (L_{PBV} - H_{PBV})\beta_{PBV} + \\ +(H_{MM} - L_{MM}) + e \quad (1.14)$$

Where $H_{MM} - L_{MM}$ is the momentum performance difference of 12 months.

1.5.4. Market Efficiency Hypothesis

The MEH theory was originally expressed by French mathematician Louis Bachelier [1900] in this thesis "The Theory of Speculation". His work was ignored until the 1950s Paul Samuelson started to get occupied with Bachelier study. Eugene Fama [1965] published his study of the random walk model and later in 1970 he published a reconsideration of the MEH theory that included the definition of the three forms[4] of the efficient market and actually established and expanded the theory. The MEH theory was widely accepted as Efficient Market

[4] Weak form, semi strong form and Strong form

Hypothesis in 1990. The MEH is one of the most basic hypotheses in finance and it was crucial in portfolio management.

1.5.5. Black, Merton and Sholes

Fisher Black and Myron Sholes [1972] developed a formula for the valuation of stock option in their research called "The pricing of options and corporate liabilities" and published in the Journal of Political, which was pioneering in the area of financial engineering. Robert C. Merton [1973] collaborated with Black and Sholes and also made publications in the matter. Merton found also another way to derive the formula and generalized in many ways. Black, Sholes and Merton put the foundation for the markets to have a rapid growth for derivatives. Their researches were essential to finance area. The French mathematician, Louis Bachelier [1900] that was mentioned earlier in the thesis also was one of the earliest attempts to value derivatives.

1.5.6. Optimization models nowadays

Since the introduction of quadratic programming by Markowitz [1952] has passed half a century. Optimization techniques, as we saw earlier in this thesis, during this period have evolved from a theoretical tool of positive analysis to a practical tool for normative analysis. Optimization models today are at the core of decision support system for financial engineers as Zenios [1993] said in his book by the title "Financial Optimization". Optimization is dominant in financial engineering[5] area. The rapid growth of companies through the years, the globalization and the rapid development of technology led us towards an enterprise wide risk management. Through the years we concluded that the risk measurement is very important for portfolio optimization and there are many factors that affect risk and the development and use of tools for measuring the risks is needed. The multi- period dynamic portfolio optimization is the best way to optimize a portfolio in an enterprise wide environment where complicated calculations and predictions of a large scale are crucial. There were many researches published the last six years and also many years earlier on this matter, Chen and Song publication is one of them. Chen and Song [2012] were involved in this optimization method in their publication called "Dynamic portfolio optimization under multi-factor model in stochastic markets". Samuelson [1969] by his paper "Lifetime portfolio selection by dynamic stochastic programming" dealt with this method many years earlier.

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[5] The mixture of theoretical finance with computer modeling to support financial decision.

Chapter 2nd

OVERVIEW OF RISK MEASURES AND PORTFOLIO IMMUNIZATION AGAINST RISK

2.1. Introduction

The most well known risk definition was given by Frank Knight [1921] as we said to the previous chapter where we analyzed the meaning of risk. In this chapter will be presented the most basic risk measures and in the end the procedure of immunization of a portfolio. As far as risk measures are concerned, mainly will be used some chapters of John C. Hull book [2012] by the title "Risk Management and Financial Institutions" and from Bertocchi, Giacommeti, Zenios[2005] research called "Risk factor analysis and portfolio immunization in the corporate bond market. Also we use some chapters from Zenios[1993] book, Financial Optimization. Finally for some of the risk measures, is used the homonymous articles of the creators.

The risk measures are categorized in the following groups:

- Statistical Risk Measures: standard deviation, volatility, variance, beta, coefficient of variation, Value at risk, Riskgrade™ Measure
- Sensitivity Factors: Duration, convexity, the greeks
- Single – scenario risk measures: Stress testing
- Risk –Adjusted performance measures: Treynor index, Sharpe index, Sortino ratio, Modigliani-Modigliani measure
- Downside risk measures: Morningstar's ranking, target shortfall, semi-variance
- Excess risk measures: Tracking error

2.2. Statistical Risk Measures

2.2.1. Variance

The term variance in Statistics is the risk measure that counts how dispersed is the probability distribution of a random variable from the expected value. The term of variation was firstly introduced by Ronald Fisher [1918] in his paper "The Correlation between Relatives on the Supposition of Mendelian Inheritance". The variance of a random variable X is symbolized by σ^2 or Var(X) and it is equal to: $Var(X) = E[(X - \mu)]^2$, where μ=E(X) is the expected value of a random variable. The variance is never negative because the squares are positive or zero. The unit of the variance is the square of the unit of the observation that is why many statisticians use instead of the variance its square root, known as standard deviation.

The variance of a portfolio, for example of two stocks, is given by the following equation:

$$\sigma^2(R_p) = x_1\,[x_1\sigma^2\,(R_1) + x_2\,Cov(R_1,R_2)] + x_2\,[x_2\,\sigma^2(R_2) + x_1\,Cov(R_1\,,R_2\,)] \quad (2.1)$$

Where: The first bracket refers to the contribution of the share 1 in the overall risk of the portfolio ($Cov(R_1,R_p)$) and respectively the second bracket refers to the contribution of the second share.

The function 2.1 could also be expressed differently as a function of the correlation coefficient of the two shares as follows:

$$\sigma^2(R_P) = x_1\,{\sigma_1}^2 + x_2{\sigma_2}^2 + 2x_1x_2\sigma_1\sigma_2\rho_{12} \qquad (2.2)$$

Where ρ_{12} is the correlation coefficient of the two shares and takes prices as follows $-1 \leq \rho \leq 1$. The smaller ρ is in absolute value the more we diversify the risk in our portfolio.

2.2.2. Standard deviation

The term standard deviation presented in statistics by Karl Pearson [1894] in the research "On the dissection of asymmetrical frequency curves". The standard deviation is symbolized by σ or std(X) and it is the positive square root of the variance. It is a measure of dispersion or variability of a random economic variable and it measures the extent to which the returns of a random variable deviate from their mean value. A low standard deviation indicates low volatility. The standard deviation represents the total investment risk, systematic and non-systematic, and is suitable as a measure when portfolios are not well differentiated and is based on the assumption of normal distribution which means there are equal variances for each side of the mean, this assumption is not satisfied in the stock markets. The standard deviation of an investment is given by the following function:

$$\sigma(R_{it}) = \sqrt{\frac{\sum_{i=1}^{T}\,(R_{it} - E(R_{it}))^2}{T}} \qquad (2.3)$$

Where R_{it} is the return of the share in a certain time period, $E(R_{it})$ is the average return of the share for this period and T is the number of periods.

Standard deviation is used to calculate any type of portfolio and gives us a direct comparison of alternative investments but it is insufficient in asymmetric risk models.

2.2.3. Coefficient of variation

The coefficient of variation measures the risk per unit of performance. It is used when the standard deviation cannot evaluate different investments in order to select the most efficient with less risk. This happens in investments that have the same risk but different expected returns or between investments with the same expected return but different risk. The coefficient of variation is given by the following equation:

$$CV = \frac{\sigma(X)}{E(X)} \tag{2.4}$$

2.2.4. Beta Coefficient

William Sharpe [1964] presented the Capital asset pricing model and the concept of beta. Beta or else systematic risk is a statistical measure that determines volatility or the risk of capital compared with a benchmark. In the U.S. published betas use a stock market index such as S&P500 as a benchmark. In Greece we have the index of Athens Stock Exchange that by definition has a beta equal to one. A fund that has a beta close to one means that its performance matches the benchmark. A beta that is greater than one indicates greater volatility than the market and a beta less than one indicates lower volatility than the market index. Investors that expect a bull market may choose funds that have a high beta, the opportunities for investors to achieve higher returns than the market is increased. On the other hand investors that expect a bear market select funds with a lot beta to achieve damage lower than the market. The beta is always compared to the market. When we have well diversified portfolios or analyze individual stocks the beta coefficient is an appropriate risk measure. Beta is given by the following type:

$$\beta_i = \frac{\sigma_{iM}}{{\sigma_M}^2} \tag{2.5}$$

Where σ_{iM} is the covariance between the return on the portfolio and the return of the market and ${\sigma_M}^2$ is the variance of the market.

2.2.5. Volatility

As Hull [2012] explains in his book:

A variable's σ, is defined as the standard deviation of the return provided by the variable per unit of time when the return is expressed using continuous compounding When volatility is used for option pricing, the unit of time is usually one year, so that volatility is the standard deviation of the continuously compounded return per year. When volatility is used for risk management, the unit of time is usually one day so that volatility is the standard deviation of the continuously compounded return per day. Define S_i as the value of a variable at the end of the day i. The continuously compounded return per day for the variable on day i is:

$$\ln \frac{S_i}{S_{i-1}} \tag{2.6}$$

This is exactly the same as: $\frac{S_i - S_{i-1}}{S_{i-1}}$

An alternative definition of daily volatility of a variable is therefore the standard deviation of the proportional change in the variable during a day. This is the definition that is usually used in risk management.

Volatilities for different units of time are basically different terms. There is no direct relationship between a weekly volatility and an annual volatility, there is an exception to this called the square root of time rule. If the fluctuations in a stochastic process from a period to the next is independent variability increases with the square root of the unit of time. Any price that follows a random walk satisfies this condition of independence. The square root of time rule is exact if volatilities are based on logarithmic returns and almost correct when are based on simple returns. Another issue is whether to use business or calendar days. In general analysts tend to ignore weekend and holidays and the usual assumption is that there are 252 days per year when we want to calculate volatilities.

Implied volatility

A way to estimate volatility for a given underlying asset is to use the value of an option on this asset. Assuming that a call option in the underlying asset is actively tradable then the price of the option is possible and can be obtained. After applying a suitable type for pricing the option, for example Black and Scholes method, we calculate the annual volatility that is included in the formula that we use to price the option. In this way we get the volatility by the price of the option and it is called implied volatility for the underlying asset.

Historical volatility

Another approach to estimate volatilities is to apply time series analysis of historical data for the variable whose volatility is to be estimated. Volatilities that are estimated that way are called historical volatilities. They are usually calculated from daily data and this means those are daily volatilities and often need to be converted in an annual base by applying the square root of time rule, especially in the case of pricing an option.

2.2.6. VaR-Value at Risk

The value at risk is a measure of risk and measures the potential loss in a portfolio for a given time period. We use it to estimate how the value of an asset or a portfolio will be declined usually in 1 or 10 days period under standard conditions. The 10 day Var does not take into account the mitigating market movements and does not express the worst result that could happen as a result of unusual and unpredictable market conditions. It expresses the maximum amount that we might lose, but only to a X level of confidence e.g. 99% and there is a certain statistical probability e.g. 1% that the damage could be greater than the estimated Var.

The Value at risk is very useful when we want to compare the risk of different portfolios. JP Morgan developed a method to calculate the Value at risk for simple portfolios, called RiskMetrics. RiskMetrics predicts the volatility of financial securities and their correlation. There are various models to estimate the Value at Risk and they have a common assumption that the best way to predict future changes is by historical data of the market. There is the variance-covariance model known as delta-normal model, the historical simulation approach that includes an operation of the current portfolio through a set of historical price changes to estimate a distribution of changes in the value of the portfolio and by calculating a percentage. Also there is the Monte Carlo simulation analysis that includes a principal components analysis of the variance-covariance model followed by a random simulation of the components. The Monte Carlo simulation analysis includes the definition of all the relevant risk factors, a construction of price scenarios and the evaluation of the portfolio for each scenario.

Expected Shortfall

Expected Shortfall is a risk measure that gives better incentives than value at risk, it is also called conditional value at risk, expected tail loss and conditional tail expectation. Although it has better properties than the value at risk, value at risk is a simpler to understand risk measure and it is easier to back-test the procedure of its calculation. Expected Shortfall belongs to coherent risk measures and encourages diversification. It is a function of two parameters, just like the value at risk, the time horizon T and the confidence level X. As Hull [2012] implies in his book expected shortfall is: *The expected loss during time T conditional on the loss being greater than the Xth percentile of the loss distribution.*

2.2.7. Riskgrade™ Measure

The Riskgrade™ is a standardized measure of variability that allows a direct comparison of investment risk against all categories of assets. The Riskgrades are indicators of risk based on volatility of returns and they are dynamic and change over time. The greater the volatility of returns, the greater is the Riskgrade of an asset. A Riskgrade equal to zero indicates that a financial instrument has no price volatility. Riskgrades are updated on a daily basis from RiskMetrics to reflect the change of the risk level for financial instruments over time. They are calculated by comparing the present estimate of financial instrument's volatility to the market-cap weighted return volatility of a different set of international capital markets, under normal market conditions.

2.3. Sensitivity Factors

2.3.1. The Greeks

There are a set of sensitivity coefficients used widely by traders to quantify risk exposures for portfolios that contain options or other derivatives. Each of these sensitivity coefficients measures how the market value of a portfolio responds to a change in a variable e.g. a value of an underlying asset, implied volatility, interest rates or time. The Greeks are:

- Delta: It measures the first order sensitivity in the price of the underlying asset.
- Gamma: It measures the second order sensitivity in the price of the underlying asset or else the degree of change of delta.
- Vega: It measures first order sensitivity to implied volatility.
- Theta: It measures the sensitivity over time.
- Rho: It measures the sensitivity in an interest rate.

<u>DELTA-price risk</u>

Delta measures the degree of change of the price of an option as a result in a small change in the price of the underlying instrument. The puts have a negative delta because they have a negative relationship to the underlying asset. On the other hand the call options have a positive relationship to the price of the underlying asset, when the price of the underlying asset increases respectively increases and the call since there are no changes in other variables. The equation of delta is the following:

$$\delta = \frac{dC}{dS} \tag{2.7}$$

Where dC represents the change in the price of the option and dS represents the change in the price of the underlying asset.

<u>GAMMA-convexity risk</u>

Gamma is the first derivative of delta and measures the rate of change of delta. It gives us information about the convexity and for its direction. The higher the gamma is, the more valuable is the option for the owner. An option that has a high gamma, when the price of the underlying security rises, it also increases the delta so that the option will be estimated more in value than in a neutral gamma position. Conversely, when the price of the underlying security declines, the delta is also reduced and the option loses less in value than in gamma neutral position.

The opposite applies in short position. Gamma is calculated by the following equation:

$$\gamma = \frac{d^2C}{dS^2} \qquad (2.8)$$

<u>THETA-time decay risk</u>

Theta is not used often by traders, it measures the extent of the decline of the premium of the put option time as a result of the passage of time. Theta is calculated by the following equation:

$$\theta = -\frac{dC}{dT} \qquad (2.9)$$

<u>VEGA-volatility risk</u>

Vega quantifies the risk of exposure to changes in implied volatility and indicates us how the price of an option will be increased or decreased given the level of implied volatility. The sellers of options benefit from a fall in implied volatility and the converse applies for the buyers. Vega is calculated by the following equation:

$$v = \frac{dC}{d\sigma} \qquad (2.10)$$

<u>RHO-discount rate risk</u>

Rho is used to measure the risk of the market in derivative portfolio. It measures the linear exposure of the portfolio to changes in the risk-free rate. Rho is calculated by the following equation:

$$\rho = \frac{dC}{dr} \qquad (2.11)$$

2.3.2. Duration

<u>Weighted duration</u>

The weighted duration measures the sensitivity of an asset or liability for changes in interest rate in a better way than the simple duration to maturity. This happens because the weighted duration takes into account the time that the cash flows are done not just the time to maturity. The weighted duration is a measure of elasticity of a security or portfolio of securities to changes of the interest rate. The larger the duration in absolute value the more sensitive is the value of a security to interest rate changes. We take the first derivative of the price P of the security as a function of the long term return and give us the following relationship:

$$\frac{\frac{dP}{P}}{\frac{dr}{(1+r)}} = -D \qquad (2.12)$$

Macaulay duration

Considering that the yield to maturity of a bond i is y_i then the Macaulay duration is:

$$D_i^{MAC} = \frac{\sum_{t=1}^{T} t \frac{F_{ti}}{(1+y_i)^t}}{\sum_{t=1}^{T} \frac{F_{ti}}{(1+y_i)^t}} \qquad (2.13)$$

Where y_i is the yield to maturity of a bond and F_{ti} are the payments or cash flows that each bond i has at time t.

The Macaulay duration is measured in units of time. The denominator is the present value of value of the cash flows of the bond discounted with the yield of the bond. The numerator is the weighted sum of the present values of the cash flows F_{ti} , where every cash flow id discounted at time t=0 from the yield of the bond and weighted by the time at which it is paid.

Modified duration

Considering a structure of interest rates by the form $r_t = r$ for all t=1,2,3…,T The modified duration of a bond i and price P_i is given by:

$$D_i^{MOD} = -\frac{dP_i}{dr}\Big/P_i = \frac{1}{P_i}\sum_{t=1}^{T} \frac{tF_{ti}}{(1+r)^{t+1}} \qquad (2.14)$$

In discrete time or by the type:

$$D_i^{MOD} = -\frac{dP_i}{dr}\Big/P_i = \frac{1}{P_i}\sum_{t=1}^{T} tF_{ti}\, e^{-rt} \qquad (2.15)$$

In continuous time where dP_i is the change in the price of the bond i, dr is the change of the interest rate, F_{ti} are the payments or cash flows of each bond i at time t, e^{-rt} is the discount rate under interest rate r in continuous time and P_i is the price of the bond i. The modified duration is measures in units of time conventionally let us consider in years.

Fischer-Weil duration

Let as assume that the structure of interest rates $\{r_t\}$ for t=1,2,…,T undergoes a parallel small change Δr $\{r_t + \Delta r\}$ for every t=1,2,…,T. The price of the bond i under the certain structure of interest rates is P_i and the sensitivity of the price in parallel change is given by the Fisher-Weil duration:

$$D_i^{FW} = -\frac{dP_i}{dr}\Big/P_i = \frac{1}{P_i}\sum_{t=1}^{T}\frac{tF_{ti}}{(1+r_t)^{t+1}} \qquad (2.16)$$

In discrete time or,

$$D_i^{FW} = -\frac{dP_i}{dr}\Big/P_i = \frac{1}{P_i}\sum_{t=1}^{T} tF_{ti}\, e^{-rt} \qquad (2.17)$$

In continuous time where dP_i is the change in the price of the bond i, dr is the change of the interest rate, F_{ti} are the payments or cash flows of each bond i at time t, e^{-rt} is the discount rate under interest rate r in continuous time and P_i is the price of the bond i.

2.3.3. Convexity

The convexity is a measure of the curvature and of how the value of the bond changes when interest rates change. Let us assume that the structure of interest rates $\{r_t\}$ for t=1,2,…,T undergoes a parallel small change Δr $\{r_t + \Delta r\}$ for every t=1,2,…,T, such that $dr_t = dr = \Delta r$ for every t=1,2,…,T. The price of the bond i under the certain structure of interest rates is P_i and the convexity is given by:

$$Q_i = \frac{\frac{d^2 P_i}{dr^2}}{P_i} = \frac{1}{P_i}\sum_{t=1}^{T}\frac{t(t+1)F_{ti}}{(1+r_t)^{t+2}} \qquad (2.18)$$

In discrete time or

$$Q_i = \frac{1}{P_i}\sum_{t=1}^{T} t^2\, F_{ti}\, e^{-tr_t} \qquad (2.19)$$

In continuous time where Q_i symbolizes the convexity of the bond i, P_i symbolizes the price at pointz_0, F_{ti} are the payments or cash flows of each bond i at time t, r_t is the instantaneous rate at time t and $\frac{d^2 P_i}{dr^2}$ is the second derivative of price to the change of the interest rate.

2.4. Single-scenario risk measures

2.4.1. Stress Testing

The stress test or scenario analysis is used as a risk measure so as to understand the changes in a portfolio under unusual market conditions. Stress tests are typical simulations which may be presented in a scenario, historical data or simulation in random sampling basis e.g. Monte Carlo analysis. Through stress testing is examined the effects of various market conditions or events in the value of a security a portfolio or a strategy. Some stress tests include how risk and return change under the use of different assumptions or models. Every portfolio manager needs to know what the weak point is in his portfolio. By determining the change in the value of its portfolio under stressful conditions the portfolio manager has a better understanding of where the risks are in his portfolio and by this way he can make transaction to reduce the risks in an acceptable level. Without stress testing when a "bad" situation happens in the market it could cause irreversible damage to the performance of the portfolio. Stress tests can "see" what could be done if the undesirable happens. The stress testing procedure should be checked again by back testing to see if the process has accurate predictions. Furthermore stress test should take under consideration all types of leverage and the related cash flows and should take place at least once every three months. Often Stress tests are presented to establish the expected exposure of the market and the credit exposure in the life of a security or securities portfolio.

2.5. Risk adjusted performance measures

2.5.1. Treynor ratio

Treynor [1965] developed the first composite measure of performance that includes risk. Treynor postulated two components, the risk generated by the fluctuations of the market and the risk resulting by the variance of the securities portfolio. He was interested in a measure that will apply to all investors regardless of their preferences in risk. The Treynor index calculates the reward of risk considered in a portfolio per unit of systematic risk.

$$T_p = \frac{R_p - R_f}{\beta_P} \qquad (2.20)$$

Where β_P is the relative risk regarding the benchmark that we have (it is the coefficient of systematic risk, the portfolio's beta),R_p is the portfolio's return and R_f is the risk free rate.

The higher the value of the Treynor ratio of a portfolio is, the better the performance of the portfolio. Also the Treynor ratio corresponding to market portfolio gives the slope of the Securities Market Line (SML). If we compare the index of a portfolio with the index of the market portfolio, then the portfolio could be presented on the same graph with the Securities Market Line. If the index of the portfolio is greater than the index of the market portfolio, the portfolio will be above the Securities Market Line, which means that during the analysis period had superior performance accordingly to its systematic risk. On the contrary if the index of the portfolio is smaller, then the portfolio will be located below the Securities Market Line, which means that during the period had lower efficiency depending on systemic risk.

2.5.2. Sharpe ratio

Sharpe [1966] in his paper of capital asset pricing model and the capital market line made the Sharpe ratio, which calculates the reward of risk of the portfolio per unit of total risk. This portfolio performance measure is similar to the measure of Treynor but it measures the overall risk of the portfolio including the standard deviation, rather than considering the systematic risk. It is a net number due to the fact that the numerator and denominator of the ratio are expressed as a percentage. The efficiency Sharpe index is expressed by the ratio:

$$S_p = \frac{R_p - R_f}{\sigma_p} \tag{2.21}$$

Where σ_p is the standard deviation of the portfolio or the individual security.

The higher the value of the Sharpe ratio of a portfolio is, the better the performance of the portfolio during the selected period. The Sharpe index or else reward-to-variability measure corresponds to the market portfolio and gives the slope of the Capital Market Line (CML). So if we compare the Sharpe ratio of a portfolio with the corresponding index of the market portfolio, then the portfolio could be presented on the same graph with the capital market line. If the index of the selected portfolio is greater than the index of the portfolio, the portfolio will be above the capital market line and that means that during the analysis period had superior performance proportionally to the overall risk. On the other hand if the ratio is small, then the portfolio will be located below the capital market line, which means that during the selected period had lower efficiency proportionally to the overall risk.

2.5.3. Jensen ratio

The measure of Jensen [1968] is similar to the two previous ratios and is based to the capital asset pricing model. Jensen's index or else Jensen's alpha is the value of alpha of a portfolio which is calculated as the difference between the realized return of the portfolio from its required return that corresponds to the systematic risk in the portfolio. Jensen's ratio is given by the following type:

$$a = R_p - R_f - \left[\left(R_M - R_f\right)\beta_p\right] \tag{2.22}$$

The Jensen ratio uses the notion of the systematic risk.

2.5.4. Sortino ratio

The Sortino ratio was made by Brian. M. Rom and named by Dr. Frank A. Sortino whose study was over downside risk optimization. This ratio measures the risk –adjusted return of an investment or a portfolio. This ratio is similar to

Sharpe's ratio but instead of the standard deviation uses the semi-standard deviation or else standard deviation of negative asset returns, which is called downside deviation. The Sortino ratio is calculated as follows:

$$S = \frac{R_p - R_f}{\sigma_d} \qquad (2.23)$$

Where σ_d is the downside semi-standard deviation of the security or portfolio of securities.

Downside semi-standard deviation is a measure of deviation of a value below a minimum acceptable price, which means that Sortino ratio penalizes only the "harmful" volatility. The upside movements are considered desirable and are not taken into account in the volatility, that helps investors t assess risk more effectively than just looking excess returns on total risk. This measure is focused on measuring how often the price of the security falls. Moreover a high Sortino ratio shows low risk in large losses to be made and it is a useful measure in cases where the returns of the portfolio are not normally distributed. In such cases the downside semi-standard deviation is a better risk measure than the standard deviation.

2.5.5. F.Modigliani-L.Modigliani measure

F.Modigliani and L.Modigliani [1997] created a new measure easier to interpret. The Modigliani –Modigliani measure is calculated by the following type.

$$M2 = S\sigma_B + R_F \qquad (2.24)$$

Where S is the Sharpe ratio,σ_B is the standard deviation of a selected benchmark portfolio and R_F is the risk free rate for the selected period.

Modigliani and Modigliani proposed the use of a broad market index such as the S&P 500 but other benchmark index can also be used for the calculation of this measure. The higher this measure is the higher the performance of the security or the portfolio of securities for any level of risk.

2.6. Downside risk measures

2.6.1. Downside volatility or Semi-variance

Downside volatility as a notion was presented by Markowitz [1959] in his paper and also by Roy in his study [1952]. The downside volatility or semi-variance is a risk measure that studies only the deviations that are below the average and is a measure that intuitively is reasonable and is used in some portfolio theories. When returns are normally distributed the downside volatility is proportional to the variance and in such cases does not provide greater insight for the risk than standard deviation. Semi-variance is of great use when the distribution of returns is asymmetric. The semi-variance is defined as follows:

$$sv = \frac{1}{n}\sum_{i=1}^{n} \min[0, (x_i - r)]^2 \qquad (2.25)$$

Where n is the sample size and r is the predetermined target rate required for the return of the security.

By expanding the notion of the semi-variance so as to by negotiating with the upside and the downside volatility of the returns of the securities then these two notions are defined as follows:

$$sv(up) = \frac{1}{n}\sum_{i=1}^{n} max[(0, (x_i - r)]^2$$

$$sv(down) = \frac{1}{n}\sum_{i=1}^{n} min\,[(0, (x_i - r)]^2$$

In the equations above n is the sample size and r is the risk-free rate. The downside volatility aims to isolate the negative part of volatility by measuring the volatility of losses and the semi-variance is a special case of downside deviation where the minimum acceptable return is the risk-free rate plus the excess risk of the investment.

2.6.2. Target Shortfall

A.D.Roy [1952] a few months after the publication of H. Markowitz [1952] "Portfolio Selection" proposed in his paper "Safety First" an alternative risk measure called target shortfall. Target Shortfall is a statistical measure that is calculated in a similar way as the downside volatility and takes into account only the observations that falls under a predetermined return. This risk measure could be used regardless the type of the distribution. The target shortfall does not provide a sufficient risk description but the evolved mean target shortfall probability vector model does. This model is suitable for practical applications and a trial shows that the average return of the model when it is used in bear markets is equal to the results of the optimization of a traditional model but thanks to the asymmetry in bull markets achieves better returns. The mean target shortfall probability vector model is regulated by a portfolio manager and maximizes the expected return subject to the constraint that portfolios must be target shortfall probability-vector feasible. The calculation is not based on the parameter of the distribution of returns as the traditional approaches but uses directly historical returns instead of parameters.

2.6.3. Morningstar's Risk

As William F. Sharpe [1998] implied in his paper "Morningstar's Risk-Adjusted Ratings" at Financial Analyst Journal, due to the rapid growth of investments through mutual funds worldwide the need to measure the performance of those funds was created. In U.S. was produced by Morningstar a risk-adjusting rating for this purpose. The Morningstar Company was the first to use an assessment method of mutual funds by the use of stars so as to evaluate the mutual fund managers. After following a certain procedure the classification is made starting from the mutual fund with higher grade and resulting to the one with the worse grade and the stars are given, 1-5 stars. Funds that after the results fall to the top 10% are given 5 stars , those in the next 22,5% get 4, in the next 35% get 3, in 22.5% get 2 and those in the bottom 10% get 1. The basic relationship which is used is to calculate the risk-adjusted rating is the following:

$$RAR_i = RRet_i - RRisk_i \tag{2.26}$$

Where RRisk$_i$ is the measure of the funds relative risk, RRet$_i$ is the measure of the funds relative return.

2.7. Excess Risk measures

2.7.1. Tracking Error

There were many researches through the years about the tracking error. Rudolf, Wolter and Zimmermann [1999] defined in their publication, called "A linear model for tracking error minimization" at the Journal of Banking and Finance the notion of tracking error. The Tracking error belongs to the excess or otherwise relative risk measures. They defined the tracking error as the volatility of excess return. Tracking error is a measure that compares how close a portfolio follows a benchmark index and it reflects all the risks associated with an active investment strategy and could be calculated using a number of different analytical measures. As far as the investors of active risk managements is concerned gives them good insight. The tracking error is based on a normal distribution and measures how close yields a financial instrument with respect to a curve. To minimize it in a portfolio the systematic risk must be as close as possible to the curve and the portfolio should be well diversified. The calculation of tracking error is as follows:

$$TE = \sqrt{Var(r_p - r_b)} \tag{2.27}$$

Where $r_p - r_b$ is the active return (the difference between the portfolio return and the benchmark return).

2.8. Portfolio immunization

The shifts of the yield curve as a result of changes in interest rates, is perceived by bond portfolio managers as a huge source of risk. In order to ensure that the bond portfolio remains unaffected by changes in interest rates they follow a particular strategy called immunization. As seen earlier, the duration is a measure of price sensitivity to a change in interest rates. The technique of immunization attempts in a sense, to eliminate the sensitivity of price to possible unexpected changes in interest rates matching the duration of the bond portfolio with the duration of the liability.

2.8.1. Necessary condition for immunization

$$\sum_{i=1}^{n} P_i X_i = P_L \qquad (2.28)$$

Where in the left side we have the present value of the assets and on the right side the value of the liabilities. Under this condition and in addition considering that the price of reinvestment rates is not changed, the cash flows of the portfolio will correspond to the cash flows of liabilities. That is an example of surplus but we would come up with the same condition even if we allowed in our model borrowing, in which case we will create a slightly different model that is written as follows:

2.8.2. First-order condition for immunization

$$\sum_{i=1}^{n} D_i{}^{FW} P_i X_i = D_L{}^{FW} P_L \qquad (2.29)$$

The following model ensures zero exposure to market risk for small and parallel shifts only in the interest rate.

Model 2.8.1 Portfolio immunization model.

Portfolio immunization model

F(x) is the objective function, P_L is the present value of the liabilities, D_i^{FW} is the Fisher-Weil duration of the i-th asset, and D_L^{FW} is the Fisher-Weil duration of the liabilities.

Maximize F(x)

Subject $\quad \sum_{i=1}^{n} P_i X_i = P_L \qquad (1)$

$\sum_{i=1}^{n} D_i{}^{FW} P_i X_i = D_L{}^{FW} P_L \qquad (2)$

x≥0

The selection issue of the objective functions needs to be examined. The objective function that is the most commonly used, is the yield of the portfolio, which is approximated by:

$$Y_P \cong \frac{\sum_{i=1}^{n} D_i{}^{FW} P_i Y_i X_i}{\sum_{i=1}^{n} D_i{}^{FW} P_i X_i} \qquad (2.30)$$

Since the denominator is constant and equal to D_L^{FW} by the constraint (2) the portfolio immunization model could be solved as a linear programming problem maximizing the linear combination $\sum_{i=1}^{n} D_i{}^{FW} P_i y_i x_i$.

To see how the basic model can be extended we consider the factors that are under our control and factors that do not seem clear from the model. The risk of interest rate from parallel shift is under our control by duration matching. Although the risk that emerges from changes in the form of interest rate is not under our control.

Let us consider an immunized portfolio structured to fund a single payment after a decade. A portfolio of matched duration can be placed with a combination of a 15 year bond and cash. If the interest rate is rising then the value of an asset portfolio will fall faster than liabilities, causing a negative net worth. Of course we will have a positive net worth if the interest rate reduced.

Another way to understand this problem is to recognize that the price-yield ratio is curved rather than linear. Therefore while for parallel shifts of the interest rates' curve, assets and liabilities move together, because of the matching duration constraint, for bigger changes assets and liabilities will diverge if they have different price-yield relationships.

We also impose a necessary second order condition so as to be sure that the price-yield curve of assets bounds from above the price-yield curve of liabilities. We did this by imposing to portfolio the following convexity constraint, where Q_L is the convexity of the liabilities.

Model 2.8.2 Portfolio immunization model under convexity constraint.

Portfolio immunization model under convexity constraint

Maximize F(x)

Subject $\sum_{i=1}^{n} P_i X_i = P_L \qquad (1)$

$$\sum_{i=1}^{n} D_i{}^{FW} P_i X_i = D_L{}^{FW} P_L \qquad (2)$$

$$x \geq 0$$

$$\sum_{i=1}^{n} Q_i P_i X_i \geq Q_L P_L \qquad (3)$$

F(x) is the objective function, P_L is the present value of the liabilities, D_i^{FW} is the Fisher-Weil duration of the i-th asset, and D_L^{FW} is the Fisher-Weil duration of the liabilities, Q_i is the convexity of the i bond and Q_L is the convexity of the liabilities.

Changes in the value of liabilities, due to the non-linearity of the price-yield relationship, are (given the validity level that offers a second-order approximation to the convexity curve) smaller than the changes in assets and therefore liabilities will continue to be funded.

The convexity constraints are also a cure for volatility risk. Large changes in the structure of interest rates will cause a collapse in first-order conditions. However by imposing on the portfolio convexity to be non-negative, we are confident that the minimum net position takes place under the current rate. Possible changes in interest rates can only increase the net position. Furthermore the convexity constraints are important, when options are included in the portfolio. Such securities may have negative convexity and it is important that the portfolio's convexity remains positive. In this case immunization models are constructed based on the option adjusted duration and the option adjusted convexity.

2.9. Factor immunization

Factor immunization is an improved immunization technique which faces the shape risk[6]. This is achieved by relaxing the assumption that interest rates are shifted in parallel, which is implied by the use of the term of duration for the optimization of the portfolio. We use a linear factor model as that which results from the factor analysis in the interest rate structure. This model represents a substantial percentage of the interest rate changes, not just the changes that occur due to parallel shifts. This sensitivity of bond prices due to changes in the factors is given by the following definition:

2.9.1. Factor modified duration

The bond price is given by:

$$P_i = \sum_{t=1}^{T} F_{ti}\, e^{-tr_t} \tag{2.31}$$

Where F_{ti} are the payments of the bond i at time t, r_t is the instant rate at time t and e^{-tr_t} is the discount rate under the interest rate r_t.

Changes in the structure of interest rates are expressed as a linear combination of k independent factors.

$$\Delta_{rt} = \sum_{j=1}^{k} B_{jt}\, \Delta f_j + \varepsilon_t \tag{2.32}$$

Where ε_t is the error which is assumed to be normally distributed and has zero mean i.e. E (ε_t) =0, Δf_j the change of the main elements, B_{jt} the weights. The

[6] It is the risk that the price of a security will change with time due to changes in the shape of the term structure of interest rates.

change in the price of the bond due to changes in the j-th factor is resulting from the total derivative of the equation 2.27.

$$\Delta P_i = -\sum_{t=1}^{T} F_{ti}\, t e^{-t r_t} \Delta_{rt} \qquad (2.33)$$

Where ΔP_i is the change in the price of the bond i, F_{ti} is the payments of bond i at time t, e^{-tr_t} is the discount rate under the interest rate r_t and Δ_{rt} is the change in interest rates.

Substituting in this expression the relation 2.28 and ignoring the disturbance term, it will give us the relationship:

$$\Delta P_i = -\sum_{t=1}^{T} (F_{ti}\, t e^{-t r_t} \sum_{j=1}^{k} B_{jt} \Delta f_j) \qquad (2.34)$$

Where ΔP_i is the change in the price of the bond i, F_{ti} is the payments of the bond i at time t, e^{-tr_t} is the discount rate under the interest rate r_t , B_{jt} the weights and Δf_j the change of the j-th factor.

The factors are independent and therefore $\frac{\partial f_j}{\partial f_{j'}}$ when j≠j'. Hence for small but not necessarily parallel interest rate shifts we conclude to the equation of the sensitivity of the bond price in regard to the changes in the j-th factor from the above equation:

$$\frac{\partial P_i}{\partial f_j} = -\sum_{t=1}^{T} F_{ti}\, t B_{jt}\, e^{-t r_t} \qquad (2.35)$$

Where $\frac{\partial P_i}{\partial f_j}$ is the derivative of the price of the bond i with respect to the j-th factor, F_{ti} are the payments of the bond i at time t, e^{-tr_t} is the discount rate under the interest rate r_t and B_{jt} the weights.

The factor modified duration of bond i with respect to the factor f_j is defined as the relative sensitivity of the weighted price by the price of the bond.

$$k_{ij} = -\frac{\partial P_i}{\partial f_j} / P_i = \frac{1}{P_i} \sum_{t=1}^{T} F_{ti}\, t B_{jt}\, e^{-t r_t} \qquad (2.36)$$

Where k_{ij} is the factor modified duration of the bond i with respect to the factor Δf_i , $\frac{\partial P_i}{\partial f_j}$ is the derivative of the price of the bond i with respect to the j-th factor, P_i is the price of the bond i, F_{ti} are the payments of the bond i at time t, e^{-tr_t} is the discount rate under the interest rate r_t and B_{jt} the weights.

We observe that if interest rates shift while we have only a single factor with weight$\beta_{1t} = 1$. Then the factor modified duration is simplified to the simple modified duration.

The independent factors are typically considered to be the key component of the interest rate correlation matrix. We do not take into account all the k basic components, but instead we calculate the first k with the largest eigenvalues until a certain percentage of interest rates will be explained.

The fact that we use only k basic components explains the disturbance term ε_t (2.28) which is absent from the following equation where all the k factors are used.

$$\Delta r_t = \sum_{j=1}^{k} \beta_{jt} \Delta f_j \tag{2.37}$$

Where Δr_t is the interest rates change, β_{jt} the weights and Δf_j the change of the j-the j-th factor.

2.9.2. Factor modified convexity

The factor modified convexity with respect to the j-th factor is defined by:

$$Q_{ij} = -\frac{\partial^2 P_i}{\partial {f_j}^2} / P_i \tag{2.38}$$

Where Q_{ij} is the factor modified convexity of the bond i, $\frac{\partial^2 P_i}{\partial {f_j}^2}$ is the derivative of the price of bond i with respect to the j-th factor and P_i is the price of the bond i.

2.9.3. Factor immunization with treasury bonds

A given change in interest rates will affect all bond prices. An increase in the interest rate of one year, will assimilate the value of any payment made in one year, no matter which bond made this payment or if a payments reflects capital or coupon. Therefore the bond prices are affected systematically by changes in the factors, which explain changes in the interest rate and it is likely to offset the influence of changes in factors.

Starting point for the formulation of a factor immunization model is the necessary condition for immunization i.e. the present value of assets is equal to the present value of liabilities.

$$\sum_{i=1}^{n} P_i x_i = P_L \tag{2.39}$$

Where $P_i x_i$ is the present value of the asset i and P_L is the present value of liabilities.

We want this equality to be maintained when factors change. By taking the first derivatives in factors we have:

$$\sum_{j=1}^{k}\frac{\partial(\sum_{t=1}^{n}P_i x_i)}{\partial f_j}df_j = \sum_{j=1}^{k}\frac{\partial P_L}{\partial f_j}df_j \quad (2.40)$$

Where $P_i x_i$ is the present value of the asset i, P_L is the present value of liabilities and df_j is the change of the j-th factor.

But since by construction the changes in factors df_j are independent, the above equation will apply to each factor.The factors df_j will be equal for each j=1, 2,…., k.

$$\frac{\partial(\sum_{t=1}^{n}P_i x_i)}{\partial f_j} = \frac{\partial P_L}{\partial f_j} \quad (2.41)$$

Where the term $\frac{\partial P_i}{\partial f_j}$ is the negative of the factor modified duration k_{ij} multiplied by the price P_i of the security, and we have the following first-order conditions for factor immunization:

2.9.4. First-order conditions for factor immunization.

$$\sum_{i=1}^{n}P_i x_i k_{ij} = P_L k_{Lj} \quad for\ all\ j = 1,2,3\ldots,k \quad (2.42)$$

Where k_{ij} is the factor modified duration of assets and k_{Lj} is the factor modified duration of liabilities, $P_i x_i$ is the present value of asset I and P_L is the present value of liabilities. By taking second derivatives in $\sum_{i=1}^{n}P_i x_i = P_L$ with respect to the factors and noting that $d^2P/df_j df'_j$ where j≠j' when the factors are independent, we have:

$$\sum_{j=1}^{k}\frac{d^2(\sum_{t=1}^{n}P_i x_i)}{d{f_j}^2}{df_j}^2 = \sum_{j=1}^{k}\frac{d^2P_L}{d{f_j}^2}{df_j}^2 \quad (2.43)$$

Where $P_i x_i$ is the present value of the asset i, P_L is the present value of the liabilities, df_j is the change of the j-th factor.

By stating the factor modified convexity of liability by Q_{Lj} we obtain the following the second- order conditions for factor immunization.

2.9.5. Second-order conditions for factor immunization.

$$\sum_{i=1}^{n}P_i x_i Q_{ij} = Q_{Lj}P_L \quad for\ every\ j = 1,2,\ldots,k \quad (2.44)$$

Where $P_i x_i$ is the present value of the asset i, P_L is the present value of the liabilities, Q_{Lj} is the factor modified convexity of liabilities and Q_{ij} is the factor modified convexity of bond i.

The following model guarantees zero market exposure against factors. If sufficient factors are selected to represent a large proportion of the variation in interest rates, then the following model creates a portfolio that has zero market exposure under many changes in interest rates.

Model 2.9.1. Factor immunization model.

Factor immunization model

Maximize F(x)

$$\text{Subject to } \sum_{i=1}^{n} P_i x_i = P_L \qquad (1)$$

$$\sum_{i=1}^{n} P_i x_i k_{ij} = P_L k_{Lj} \quad \text{For all j=1, 2,....,k} \qquad (2)$$

$$x \geq 0$$

Where F(x) is the objective function that we maximize, $P_i x_i$ is the present value of assets i, P_L is the present value of liabilities, k_{Lj} is the factor modified duration of liabilities and k_{ij} is the factor modified duration of assets.

In this model we could enforce convexity constraints to obtain a portfolio that satisfies the second-order condition: This constraint is imposed as an inequality.

$$\sum_{i=1}^{n} P_i x_i Q_{ij} \geq Q_{Lj} P_L \ \ for\ all\ j = 1,2, \dots k \qquad (2.45)$$

Where $P_i x_i$ is the present value of asset i, P_L is the present value of liabilities, Q_{Lj} is the factor modified convexity of liabilities and Q_{ij} is the factor modified convexity of bond i.

The reason that we allow greater factor convexity in assets than liabilities is simple. For bonds the convexities are positive. The inequality constrain imposes a non-negative net portfolio convexity with zero net duration in current prices. If a factors change the net value of portfolio will be positive. Caution is required to the second-order constraint for securities with negative convexity. The immunization models for securities with embedded options and negative convexity are based in option adjusted duration and option adjusted convexity.

2.10. Factor immunization for corporate bonds.

The factor immunization techniques could be extended so as to offset changes in the yields of corporate bonds, except from the interest rate and the shape risk. Risk analysis in the corporate bond market is an important issue in risk management. We could invoke the immunization factor model we mentioned above in order to analyze the structure of yields on corporate bonds. However in the case of corporate bonds we handle multiple credit rating classes.

The yields are correlated both between bonds with different maturities, but also in bonds with different credit ratings. Table 2.1 shows the correlation between the yields on bonds of different maturities, with ratings from AAA to B3. We must be careful for example the high correlation of yield on AAA and B3 bonds with 10-year maturity, but also the significant correlation between the 10-year AAA and the 6-months B3. These correlations must be taken into serious consideration in an effective immunization strategy.

In this part we will develop first a model which treats the rating of the bonds as independent. We will notice that this model has some advantages. We will develop a model immediately after which immunizes against changes in yields of multiple correlated evaluative rankings.

Table 2.1: Weekly yields correlations for AAA bonds and B3 from March 1992 to June 1999.

	6M AAA	1Y AAA	2Y AAA	5Y AAA	7Y AAA	10Y AAA	6M B3	1Y B3	2Y B3	5Y B3	7Y B3	10Y B3
(AAA)												
6M	1.00											
1Y	0.96	1.00										
2Y	0.93	0.97	1.00									
5Y	0.92	0.94	0.98	1.00								
7Y	0.94	0.93	0.97	0.99	1.00							
10Y	0.90	0.93	0.96	0.98	0.99	1.00						
(B3)												
6M	0.62	0.58	0.57	0.55	0.54	0.55	1.00					
1Y	0.63	0.64	0.63	0.61	0.60	0.60	0.98	1.00				
2Y	0.72	0.73	0.74	0.73	0.72	0.71	0.93	0.96	1.00			
5Y	0.82	0.84	0.87	0.88	0.87	0.87	0.75	0.80	0.89	1.00		
7Y	0.81	0.83	0.85	0.88	0.88	0.88	0.65	0.71	0.81	0.96	1.00	
10Y	0.83	0.85	0.86	0.88	0.88	0.89	0.66	0.70	0.80	0.95	0.96	1.00

Source: May 2005 Research quarterly published by the Bond Market Association, NY and www.bondmarkets.com

2.10.1. Factor immunization with uncorrelated credit ratings.

Changes in the factors of the yield curve of a rating class of a bond affect systematically the bond prices in this rating class. It is therefore likely to offset the influence of the changes in factors. We start with a model to hedge uncorrelated yield changes of a rating class and then continue with a model that will handle the existence of correlations.

Suppose C is the number of the rating classes and suppose that c represents the c-th credit rating. The parameters in the fair value equation of the bond are:

$$P_i^c = \sum_{t=1}^{T} F_{ti}^c e^{-tr_t^c} \quad (2.46)$$

Where P_i^c is the price of bond i with c-th credit rating, F_{ti}^c are the payments of bond i of c-th credit rating at time t, $e^{-tr_t^c}$ is the discount rate under the interest rater_t^c.

A linear model for the changes in yield of a bond with c rating takes the following form:

$$\Delta y_t^c = \sum_{j=1}^{k} \beta_{jt}^c df_j^c + \varepsilon_t \qquad (2.47)$$

Where we assume that k independent factors explain the yield changes of each class, β_{jt}^c are the weights of each factor, df_j^c is the change in factors of bond of class c and ε_t is the disturbance term. We assume that k independent factors explain in interest rates and are sufficient for all classes. Of course factors will be typically different for each class, but about 3-4 factors are usually sufficient to explain a significant percentage of changes in yields in each class. The sensitivity of bond prices to changes in the factors is given by:

$$\frac{\partial P_i^C}{\partial f_j^c} = -\sum_{t=1}^{T} F_{ti}^c \beta_{jt}^c te^{-tr_t^t} \qquad (2.48)$$

Where $\frac{\partial P_i^C}{\partial f_j^c}$ is the change in the price of the bond i of class c with respect to the change of j factor, F_{ti}^c are the payments of the bond i and of c-th credit rating at time t, $e^{-tr_t^t}$ is the discount rate under the interest rate r_t^c and β_{jt}^c are the weights of each factor.

The factor modified duration of the bonds with respect to the j-th factor is the relative sensitivity of the prices:

$$k_{ij}^c = -\frac{\partial P_i^c}{\partial f_j^c} / P_i^c = \frac{1}{P_i^c} \sum_{t=1}^{T} F_{ti}^c \beta_{jt}^c te^{-tr_c^t} \qquad (2.49)$$

Where $\frac{\partial P_i^C}{\partial f_j^c}$ is the change in the price of the bond i of class c with respect to the change of j factor, F_{ti}^c are the payments of the bond i and of c-th credit rating at time t, $e^{-tr_t^t}$ is the discount rate under the interest rater_t^c, β_{jt}^c are the weights of each factor and P_i^c is the price of bond i with c-th credit rating.

Let us now consider an asset portfolio with holdings x_i^c of the bond i with credit rating c, and let us also assume for convenience the same number of bonds in each class, let it be n. The necessary condition for immunization is:

$$\sum_{i=1}^{c} \sum_{i=1}^{n} P_i^c x_i^c = P_L \qquad (2.50)$$

Where x_i^c are the holdings of the bond i with credit rating c, P_i^c is the price of bond i with c-th credit rating and P_L is the present value of liabilities.

This condition should apply when the factors are changed. We take derivatives in factors from both members of the equation and we demand the sensitivities of both members for each factor f_{jc} to be equal in order to reach in a necessary first-order condition.

$$\sum_{i=1}^{n} k_{ij}^c P_i x_i^c = k_{ij}^c P_L \tag{2.51}$$

For all j=1,… k ^{c} and c=1,2,….C

Where k_{jL}^c is the weight of factor j for the liabilities of the credit rating c, k_{ij}^c is the weight of the factor j for the assets of the credit rating c, x_i^c are the holdings of the bond i with credit rating c, P_i^c is the price of bond i with c-th credit rating and P_L is the present value of liabilities.

Let us assume that the liability has a rating C=1, then the optimization model will be written as follows:

Model 2.10.1. Immunization model with uncorrelated credit ratings

Immunization model with uncorrelated credit ratings

Minimize F(x)

Subject to $\sum_{c=1}^{C} \sum_{i=1}^{n} P_i^c x_i^c = P_L$

$$\sum_{i=1}^{n} k_{ij}^c P_i x_i^c = \begin{cases} k_{jL}^1 P_L & for\ all\ j = 1, \dots k, c = 1 \\ 0, & for\ all\ j = 1, \dots k, c = 2, \dots, C \end{cases}$$

x≥0

where F(x) is the objective function that we maximize, k_{jL}^c is the weight of the factor j for liabilities with credit rating c, k_{ij}^c is the weight of factor j for assets with credit rating c, x_i^c are the holdings of the bond i with credit rating c, P_i^c is the price of bond i with c-th credit rating and P_L is the present value of liabilities.

The constraints impose that x_i^c=0 for every i when c≠1. So the holdings in assets with credit ratings different from the target c=1 are excluded. These constraints can be relaxed so as to allow some exposure to credit risk:

$$\sum_{i=1}^{n} k_{ij}^c P_i^c x_i^c = k_{jL}^1 P_L \quad for\ c = 1 \tag{2.52}$$

$$\underline{k^c} \le \sum_{i=1}^{n} k_{ij}^c x_i^c \le \overline{k^c} \quad for\ all\ c = 2, \dots, C \tag{2.53}$$

Where $\underline{k^c}$, $\overline{k^c}$ are respectively the minimum and maximum amounts of bonds that our model allows us to hold in order to have some exposure to credit risk, k_{jL}^c is the weight of the factor j for liabilities with credit rating c, k_{ij}^c is the weight of factor j for assets with credit rating c, x_i^c are the holdings of the bond i with credit rating c, P_i^c is the price of bond i with c-th credit rating and P_L is the present value of liabilities.

The parameters $\underline{k^c}$ and $\overline{k^c}$ limit the exposure to credit risk factors that affect credit classes different from the target credit class for which the risk has been hedged.

2.10.2. Factor immunization with correlated credit ratings.

We examine the case where the credit classes are correlated and let us consider an immunization model. To determine the factors that affect the yield curve of yields of different ratings classes we perform a basic data analysis to correlation table we see in Table 2.1. The upper left and lower right triangular submatrix are correlations of changes in yields of securities within the same credit rating, while the lower left square submatrix is the correlation matrix of the yields of securities within different ratings. When we assume that the rating classes are uncorrelated, like we did in the previous section, data analysis is performed for each triangular submatrix independently.

For the model that we will now study the analysis is performed in the overall table. This analysis identifies k factors that jointly affect changes in bond yields of different maturities for all classes. Assuming that x_i^c symbolizes the factor modified duration of bond i with credit rating c of the j-th factor and that K_{Lj} symbolize the factor modified duration of liabilities we have the following model:

Model 2.10.2. Immunization model with correlated credit ratings.

Immunization model with correlated credit ratings

Minimize F(x)

Subject to $\sum_{c=1}^{C} \sum_{i=1}^{n} P_i^c x_i^c = P_L$

$\sum_{i=1}^{n} k_{ij}^c P_i^c x_i^c = k_{Lj} P_L$ For all j=1, 2,..K, c=1, 2,…C

x≥0

where F(x) is the objective function that we maximize, k_{jL}^c is the weight of the factor j for liabilities with credit rating c, k_{ij}^c is the weight of factor j for assets with credit rating c, x_i^c are the holdings of the bond i with credit rating c, P_i^c is the price of bond i with c-th credit rating and P_L is the present value of liabilities.

Chapter 3rd

REVIEW OF PORTFOLIO OPTIMIZATION WITH STOCHASTIC PROGRAMMING

3.1. Introduction

In this chapter we examine optimization of dynamic trading strategies. Investors dynamically balance their portfolio at some distinct trade days in the future, based on the information. At first we will do an overview of some simple decision rules to balance portfolios. Then we will study the stochastic commitment, which is formulated as a simple model for the optimization of dynamic strategies of short borrowing and lending decisions. Finally linear stochastic programming is presented as a flexible tool for the development of a wider range of financial models.

3.2. Background for Dynamic models.

Many financial decision-making problems include liability flows expanded in the future. For example, the planning horizon for most insurance products extends over a decade, for the pension funds is more than thirty years and for social security plans may be up to fifty. It is reasonable under these circumstances to be investigated dynamic strategies that allow portfolio managers to balance their portfolio in some discrete negotiation dates in the future, based on new information. The use of discrete time is most appropriate for modeling dynamic strategies.

A dynamic strategy is a sequence of decisions of purchases and sales, including short-term borrowing and lending. During the execution of such a strategy, the manager must pay transactions cost, faces credit constraints and a difference between borrowing interest rates and lending interest rates. Constraints may also be imposed on the composition of the portfolio due to regulatory restrictions or of the corporate policy. The decisions concerning the portfolio can be made in a finite number of points in time which are called transaction dates and they extend from today until the last day before the end of the planning horizon. We assume that nothing happens between the trade dates, i.e. there are no decisions on portfolio investments and there are no payments of coupons or dividends.

3.3. Lattice structures

The prices of securities and liabilities at initial time t=0 are known, but future prices and future liabilities are unknown. We assume that at each future date of negotiation a finite number of economic conditions is possible. In every states of the economy are defined uniquely the prices of the securities and the value of the liabilities. A binomial lattice, as illustrated in Figure 3.1. shows a state of the economy at different transaction dates. The nodes in the lattice represent states of the economy and the connections (branches) represent probability transitions between economic situations. The term binomial shows that only two transitions are possible at the next time period, when someone starts from any given state of the economy he moves either in an upward or a downward state of the economy. The term lattice shows that the transitions are reconnected and an "upstream" movement is followed by a "downward" movement and leads in the same state of the economy such as a "downward" movement is followed by an "upward" movement. As a result of this property a given state of the economy can be achieved by a number of states of the economy in previous trade dates. The state of the economy in a lattice is denoted by s and the sample space of possible states of the economy at negation time t is denoted by $\sum_t$.

3.3.1. Linear scenario structure

A path between states of the economy from t=0 until T is a sequence $(s_0, s_1, \ldots . s_T) \in \sum_0 \times \sum_1 \times \ldots \ldots \sum_T$. Such a sequence of states of the economy is a scenario and is denoted by l. Every state of the economy in $\sum_t$ cannot be achieved by any state of the economy in $\sum_{t-1}$, the previous period. For example in the binomial lattice in Figure 3.2 we observe that s_2^0 can be achieved by s_1^0 but no by s_1^1. Given a state of the economy s in trading time t<T, we denote by s^+ the set of all states of the economy that can occur with positive probability in time t+1. These are called the successor states of s. Contrariwise each state of the economy s at time t>0 can be achieved by at least one state of the economy at time t-1, called the predecessor s and is denoted by s^-. Starting from a given s at time t-1 a scenario l visits only one successor s at time t and that s is denoted by $n_t(l)$. For example, in a scenario l that is described by the sequence $(s_0, s_1, \ldots . s_T)$ we have that $n_0(l) = s_0$, $n_1(l) = s_1$, $n_2(l) = s_2$ and so on. The time indicator of n will be forfeited when there is no ambiguity.

Scenarios may have common states of the economy until a certain trading date in the future. For example, the paths $(s_0^0, s_1^0, s_2^0, s_3^0)$ and $(s_0^0, s_1^0, s_2^1, s_3^1)$ from our example, they have common states of the economy until the trading date t=1. The scenarios in Figure 3.2 are independent of one another, except from the common initial state s_0^0. The set of scenarios that ignore the information about the common features of the states of economy are a scenario tree.

Figure 3.1: Recombining binomial lattice with three trading dates.

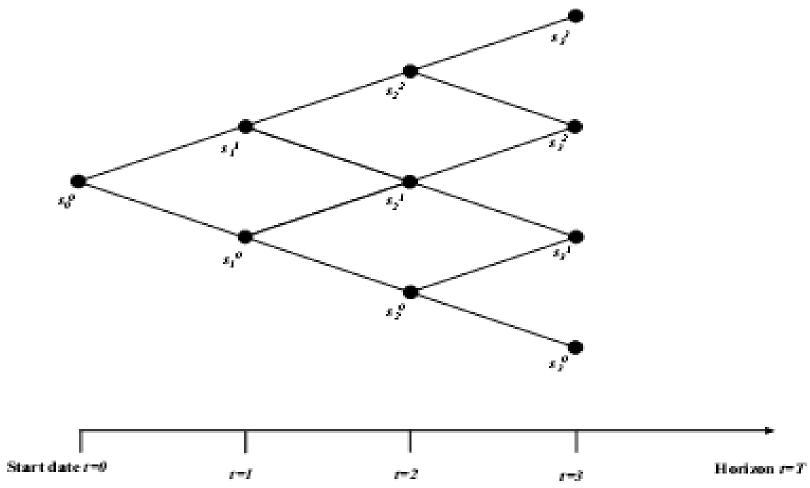
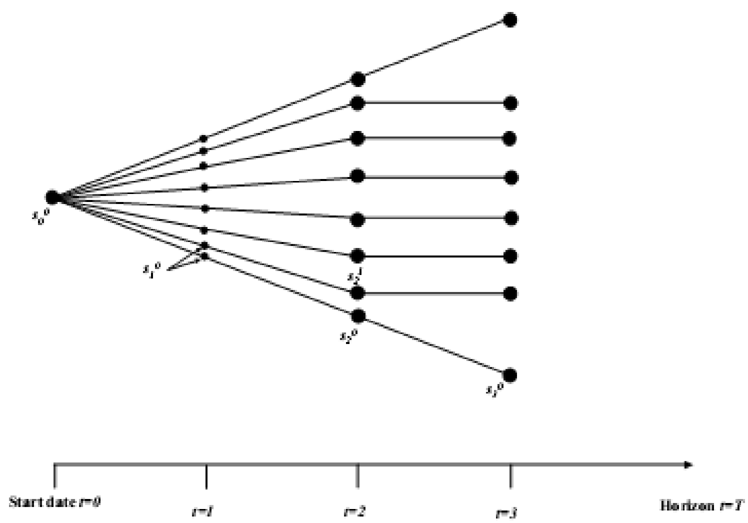


Figure 3.2: Linear scenario structure (non recombining) drawn from a binomial lattice with three trading dates.



Source: Chapter 6 of Zenios S. (1993), Financial Optimization, Cambridge University Press

The scenario optimization models that are based on non reattached scenarios allow only one trading date at time t=0 and trading strategies are single-period. To model dynamic trading strategies in a linear scenario structure we come across a dilemma. Either we do not allow transactions in periods t>0, excluding short-term cash flows from borrowing to cover deficits and surpluses where we allow transactions in periods t>0, which can vary between scenarios even with common states of the economy. The first option is clearly restrictive, since it does not provide to investors the flexibility of adjusting their portfolio as new information becomes available. They can only react to new information in short-term borrowing or lending. The second option is to relax the real problem. In fact trading strategies cannot depend on what happens in the future, and when the two scenarios share the same history up to the time t, the optimal trading strategy until this time must be identical for both paths. The decisions that are not the same until the time t violate the logical requirement for absence of clairvoyance.

3.4. Event trees

Trading strategies that satisfy the logical requirement for independence from hindsight are called non-anticipative. To model non-anticipated dynamic strategies we define scenarios on an event tree, see Figure 3.3. Information on commonality of states of the economy are contained in the structure of the event tree and the states of economy store new information that arrive at the corresponding trading date.

An event tree can be represented formally as a directed graph $\mathcal{G} = (\Sigma, \mathcal{E})$ where notes Σ denotes the time and the connections (or sectors) $\mathcal{E}$ indicate possible transitions between economic conditions, as time progresses. By the time the state of the economy represented by $\Sigma_t = \{s_t^v | v = 1,2, \ldots\ldots S_t\}$, where S_t is the number of possible states of the economy at time t. Therefore $\sum = \cup_{t=0}^{T} \sum_t$ and $\mathcal{E} \subset \sum \times \sum$. The data of $\mathcal{E}$ are represented by the ordered pairs $(s_t^{v(t)}, s_{t+1}^{v(t+1)})$ where we explicitly indicate the dependence of the index v on t. The order of the nodes indicates that state $s_{t+1}^{v(t+1)}$ at time t+1 can be reached from state $s_t^{v(t)}$ at time t. $s_{t+1}^{v(t+1)}$ is the successor state and $s_t^{v(t)}$ is the predecessor state. That is, $s_t^{v(t)+} = s_{t+1}^{v(t+1)}$ and $s_{t+1}^{v(t+1)-} = s_t^{v(t)}$.

An event tree has the following properties: (i)$\sum_0 = \{s_0^0\}$ and s_0^0 is known as the root node and it has no predecessor. (ii) Every state of the economy $s_t^{v(t)}$ has a unique predecessor from the set of states $\sum_{t-1}$ for all t=1,2,…T. The uniqueness of predecessors implies that the graph $\mathcal{G}$ has no cycles.

3.5. Scenarios

A scenario is a path of the graph $\mathcal{G} = (\Sigma, \mathcal{E})$ represents an event tree and denoted by the sequence $\left\{s_0^{v(0)}, s_1^{v(1)}, \dots s_{\tau_1}^{v(\tau_1)}\right\}$ such that, $s_t^{v(t)}$, $s_{t+1}^{v(t+1)}$ $\in \mathcal{E}$ for all t=0,1,…τ_l, $\tau_l < T$ where τ_l is the last trading date at scenario l with relative probability $p_l \geq 0$. Every scenario is denoted by l from a sample set Ω and the probabilities satisfy $\sum_{l \in \Omega} p^l = 1$.

In general we have that $\tau_l < T$ for all scenarios. However, when we handle securities that can fail, the default time is a random variable and the use of time horizons depends on the scenarios that are essential. This view also applies to cases of securities with embedded American type rights (American type options).

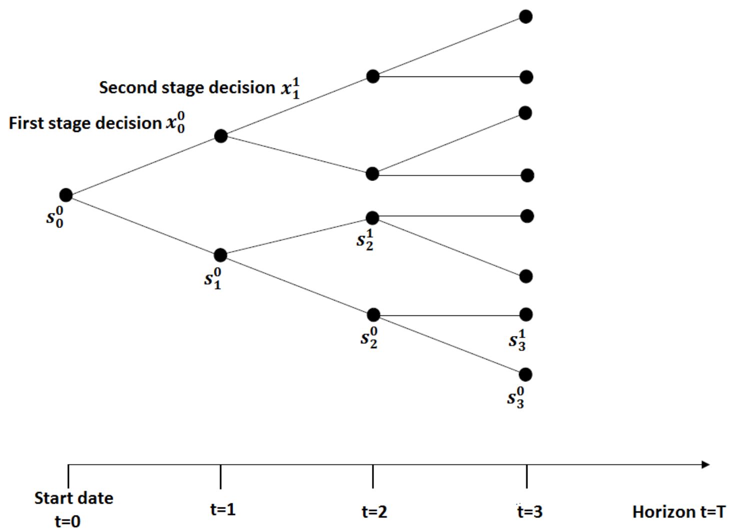
The definition of scenarios by the use of event trees clearly shows that some scenarios may have a common state of the economy until a given trading date. For example in Figure 3.3 we observe that four scenarios share common state of the economy s_0^0, s_1^0 until time t=1, while four others shares$s_0^0 . s_1^1$. A dynamic trading strategy for the first four scenarios may differ in trading dates t>1, but the portfolio that balances the time t=1, must be the same for these scenarios as they are of the same state of the economy s_1^1. In an event tree it is clear that the states of economy share s_2^0, s_2^1 the same predecessor.

We will consider dynamic strategies that use both linear scenario structures and event tress. The control decision variables for dynamic strategies on a scenario tree increase linearly with the number of trading dates. The size of the dynamic models in event trees grows exponentially with the number of dates. The strategies developed in linear scenarios are easy to calculate, but there are simplifications that lead to sub-optimal decisions. Strategies to event trees are computationally difficult to estimate, but lead to optimal decisions.

We will consider three different approaches for modeling dynamic strategies. First we will introduce briefly decision rules for the determination of dynamic strategies, soon after we will formulate stochastic models which will be an extension of the known deterministic dedication models. These models optimize a dynamic borrowing or lending strategy with cash in a linear scenario tree without foreseeing rebalancing of the portfolio.

Finally we will develop models of dynamic strategies with event trees by using stochastic programming. These models lead to the best strategies by taking into account the possibility of rebalancing the portfolio.

Figure 3.3: Event tree drawn from a binomial lattice with three trading dates.



Source: Chapter 6 of Zenios S. (1993), Financial Optimization, Cambridge University Press

3.6. Decision rules for Dynamic Portfolio Strategies.

The dynamic portfolio strategies can be determined by simple rules. As more information arrives in a linear scenarios structure, portfolios are balanced according to the rules. There is nothing optimal about dynamic strategies that are based on decision rules, but it is easy to be defined and quantified and work well under some circumstances. The decision rules are also intuitively appealing, since it is easy to be spread to portfolio managers and be applied in practice. Additionally they are useful in the investigation of issues that surround the dynamic optimization strategies and often are used as reference points with which are compared the optimized dynamic strategies. We consider four decision rules:

1. Buy- and- hold.
2. Constant mix.
3. Constant proportion.
4. Option based portfolio insurance.

3.6.1. Buy-and-hold strategy

A buy-and-hold strategy determines the percentage of initial wealth invested in the risk-free rate, and the rest invested in an asset with risk at time t=0. This portfolio is held to maturity under all scenarios, so there is nothing dynamic regarding this strategy except that the portfolio value varies depending on the scenarios.

By assuming an initial wealth of V_0 the portfolio takes the value:

$$V_{pt}^{s} = V_0 x_0 + V_0(1 - x_0)I_{t-1}^{s} \quad (3.1.)$$

Where V_{pt}^{s} is the portfolio value at trading time t in the state of economy s, V_0 is the initial wealth, x_0 is the percentage of the initial wealth that we invested in the risk-free asset, $1 - x_0$ is the percentage of the initial wealth that we invested at the asset with risk and I_{t-1}^{s} is the total return of the market from the previous state of the economy s^{-} at trading date t until the state s at trading date t-1.

3.6.2. Constant mix strategy.

A constant mix strategy that determines the percentage of the asset without risk V_{ft}^{s} and the asset with risk V_{lt}^{s} that regarding the total wealth of the portfolio should remain stable in all trading dates t and at all states of the economy s. As the market index fluctuates, the portfolio should be balanced so that the mix of assets with risk and without risk to remain stable. In particular, if the market index decreases, the portfolio is balanced by the sale of assets without risk and market risk assets, maintaining the exposure of the portfolio index. Conversely, if the market index rises, then the risky assets are sold and purchased as assets without risk, reducing exposure to risk index to its original level. This is truly a dynamic strategy, the portfolio is balanced to reflect changes in the market.

The initial value of the portfolio is given by:

$$V_0 = V_0 x_0 + V_0(1 - x_0) \quad (3.2)$$

Where V_0 is the initial wealth, x_0 is the percentage of the initial wealth that we invested in the risk-free asset, $(1 - x_0)$ is the percentage of the initial wealth that we invested in the asset with risk.

At some future trading date t and given the state of the economy s the parts of the portfolio with and without risk are given by:

$$V_{ft}^{s} = V_{p(t-1)}^{s-} x_0 \quad (3.3)$$

$$V_{lt}^{s} = V_{p(t-1)}^{s-}(1 - x_0)I_{t-1}^{s}) \quad (3.4)$$

Where V_{ft}^{s} is the value of the portfolio due to investment in risk-free assets, $V_{p(t-1)}^{s-}$ is the total value of the portfolio at the previous trading date t-1 at the previous state of economy s^{-}, x_0 is the percentage of the initial wealth that we invested in the risk-free asset, $(1 - x_0)$ is the percentage of the initial wealth that we invested in the asset with risk, I_{t-1}^{s} is the total return of the market from the previous state of the economy s^{-} at trading date t until the state s at trading date t-1, V_{lt}^{s} is the value of the portfolio due to investment in assets with risk.

However if the market index deviates from unit, the portfolio should be balanced so that the parts of the portfolio without risk and with risk to be given respectively by the formulas:

$$V_{ft}^{s} = V_{pt}^{s} x_0 = (V_{p(t-1)}^{s-} x_0 + V_{p(t-1)}^{s-}(1-x_0) I_{t-1}^{s}) x_0 \qquad (3.5)$$

$$V_{It}^{s} = V_{pt}^{s}(1-x_o) = \left(V_{p(t-1)}^{s-} x_0 + V_{p(t-1)}^{s-}(1-x_o) I_{t-1}^{s}\right)(1-x_0)) \qquad (3.6)$$

It is also easy to see that by this balance the percentage of the value without risk to the overall value remains stable from period t until t-1.

3.6.3. Constant proportion strategy.

This strategy identifies a fixed percentage of assets invested in risky assets, and as the value of assets of the portfolio changes, the portfolio is balanced so that the proportion of unsafe assets to remain constant. In fact a constant proportion strategy provides a lower bound, below of which is not allowed to fall the value of assets.

Therefore, if indicated by g the lower bound, this strategy is described by the following relationship which must be valid for each time period and for each state of the economy s: where by V_{pt}^{s} is represented the value of the assets of the portfolio.

$$V_{It}^{s} = \mu(V_{pt}^{s} - \underline{g}) \qquad (3.7)$$

Where $\underline{g}$ is the lower bound, V_{pt}^{s} is the total value of the assets of the portfolio, V_{It}^{s} is the value of the portfolio due to investing in assets with risk and μ is a constant.

It is worth noticing that a buy-and-hold strategy is a special case of a constant proportion strategy, if $\mu = 1$ and the lower bound is equal to the initial investment in the safe asset. The constant proportion strategies are also special cases, where the lower bound is equal to zero and μ varies.

In the case of constant proportion strategies as the value of assets of portfolio reduces the assets with risk are sold. Conversely, as the value of the portfolio increases the exposure to assets with risks increases too. The portfolio can respond quite well in the lower bound, even in harsh fall in the market, unless if the market fall happens so fast that the portfolio cannot keep the balance .As long as the market drop is less than $1/\mu$ the barrier will be achieved. It is worth pointing out that if a large volume of investors or investors with large positions follow this strategy, their actions will worsen the decline of an already declining market.

3.6.4. Option based portfolio insurance.

This strategy deals with achieving a target floor (lower bound) for the portfolio at the end of the investment horizon. This target implies a floor for each prior period discounted at the safe rate. These strategies lay out a portfolio of safe and risky assets so that their returns to match with those of a portfolio that includes safe assets and call options. In particular, the safe asset is kept equal to the floor, while any excess value over the floor is invested in call options.

3.7. Stochastic dedication.

We will now consider a model of optimizing dynamic strategies. The stochastic dedication model that we are going to develop optimizes short term borrowing and lending decisions as new information arrive, but does not allow rebalancing of a portfolio. Decisions at time t=0 are optimized with short term borrowing and lending decisions in future trading dates. The optimization takes place on a linear scenario tree. We can view stochastic dedication as an optimized buy-and-hold strategy with a dynamic component of borrowing and lending. The model is an extension of portfolio dedication models with the addition that incorporates the uncertainty of values and cash flows through scenarios. In this way the stochastic dedication extends the scope of classical portfolio dedication further than achieved by portfolio immunization.

The starting points for the development of the model are the necessary conditions for immunization, which we generalize for the case where the value of the assets and liabilities depend on the scenario. Considering possible states of the economy of short term interest rates r_{ft}^{s}, s $\in \sum_t$ in trading periods t $\in$ T we define the discount factors by d_T^1:

$$d_T^l = \prod_{t=0}^{T} \frac{1}{1+r_{ft}^{n(l)}} \qquad (3.8.)$$

Where n(l) denotes the state of the economy that visited the path l at time t, t is ranging from 0 to τ and $r_{ft}^{n(l)}$ is the short term interest rate under n(l). The present value of asset i in scenario l is given by:

$$P_{0i}^l = \sum_{t=1}^{T} d_t^1 F_{ti}^{n(l)} \qquad (3.9.)$$

Where $F_{ti}^{n(l)}$ are the payments of the asset I at time t under n(l) and d_t^1 are the discount factors as defined earlier. The expected present value of assets I is given by $P_{0i}^{l} = \sum_{l \in \Omega} p^{l} P_{0i}^{l}$ and it is equal to the implied market price P_{0i}^{l} with zero option adjusted spread.

The present value of liabilities at scenario l is given by:

$$P_L^l = \sum_{t=1}^{T} d_t^l L_t^l \qquad (3.10)$$

Where d_t^l are the discount factors as defined earlier and L_t^l is the payments of liabilities.

We note that the same discount factors are used for both assets and liabilities. The implicit assumption is that the short term borrowing rate to finance deficits and the short term lending rate to invest the surplus is the same. This case will be loosening latter in the study. In order for the portfolio to remain loyal in all the scenarios, we impose an immunization condition for each scenario as follows:

3.8. Necessary condition for scenario immunization

$$\sum_{i=1}^{n} P_{0i}^{l} x_i = P_L^l \quad for\ every\ l \in \Omega \qquad (3.11)$$

Where P_{0i}^{l} is the present value of the asset i under the scenario l at time 0, x_i is the holding of asset i and P_L^l is the present value of liabilities under scenario l.

Under this condition, the current value of the portfolio of assets is equal to the present value of liabilities in all scenarios. Assuming unlimited borrowing and lending in short term interest rates the portfolio remains committed under all scenarios. This condition may be impossible to meet for all scenarios. Instead of seeking a solution that applies all scenarios for the stochastic dedication model, when the portfolio of assets exceeds liabilities, against risk when the portfolio underperforms. The present value of the portfolio of assets and liabilities, with asset holdings x_0 at time t=0 and exposure of liabilities $L^l = (L_t^l)_{t=0}^{T}$ at scenario l is given by:

$$V(x; P^l) = \sum_{i=1}^{n} P_{0i}^{l} x_{0i} - P_L^l \qquad (3.12)$$

Where $V(x; P^l)$ is the present value of the portfolio of liabilities and assets, P_{0i}^{l} is the present value of asset I under scenario l at time 0, x_{0i} is the holding of asset I at time 0 and P_L^l is the present value of liabilities under scenario l.

Model 3.1: Minimization model of initial cost (Stochastic dedication)

Minimization model of initial cost

Minimize u_0

Subject to $\sum_{i=1}^{n} F_{0i} x_{0i} + u_0 + u_0^{-0} = L_0^0 + u_0^{+0}$

$$\sum_{i=1}^{n} F_{(t-1)i}^{s} x_{0i} + \left(1 + r_{f(t-1)}^{s}\right) u_{t-1}^{+s^{-}} + u_t^{-s} = L_t^s + u_t^{+s} + (1 + r_{f(t-1)}^{s} + \delta) u_{t-1}^{-s^{-}}$$

For all t=1, …, T s∈ $\sum_t$,

$$x, u^+, u^- \geq 0$$

Where u_0 is the initial cost of the investment, x_{0i} is the holding of the asset i at time 0, F_{0i} are the payments of asset i at time 0, L_0^0 is the present value of liabilities at time 0 and $r_{f(t-1)}^{s}$ is the short term interest rate at time t-1 under scenario s.

We must explicitly set $u_T^{-s} = 0$ to avoid borrowing at the last period, since the model has no mechanism to re-pay loans pending at time T.

The model follows closely the formulation of the classical portfolio dedication model. The first equation $\sum_{i=1}^{n} F_{0i} x_{0i} + u_0 + u_0^{-0} = L_0^0 + u_0^{+0}$ is the accounting equation of cash flows at time t=0 and the equations $\sum_{i=1}^{n} F_{(t-1)i}^{s} x_{0i} + \left(1 + r_{f(t-1)}^{s}\right) u_{t-1}^{+s^{-}} + u_t^{-s} = L_t^s + u_t^{+s} + (1 + r_{f(t-1)}^{s} + \delta) u_{t-1}^{-s^{-}}$ for all t=1, …, T s∈ $\sum_t$, match cash flows from assets with those of liabilities in future trading date and in all states of economy. A similar formulation maximizes the expected horizon return.

Model 3.2: Expected horizon return model.

Expected horizon return model

Maximize $\sum_{l \in \Omega} p^l u_T^{+s}$

Subject to $\sum_{i=1}^{n} F_{0i} x_{0i} + u_0 + u_0^{-0} = L_0^0 + u_0^{+0}$

$$\sum_{i=1}^{n} F_{(t-1)i}^{s} x_{0i} + \left(1 + r_{f(t-1)}^{s}\right) u_{t-1}^{+s^{-}} + u_t^{-s} = L_t^s + u_t^{+s} + (1 + r_{f(t-1)}^{s} + \delta) u_{t-1}^{-s^{-}}$$

For all t=1, …, T s∈ $\sum_t$,

$$x, u^+, u^- \geq 0$$

Where u_T^{+s} is the final surplus, p^l is the possibility if a l scenario to happen, x_{0i} is the holding of asset i at time 0, F_{0i} are the payments of asset i at time 0, L_0^0 is the present value of liabilities at time 0 and $r_{f(t-1)}^{s}$ is the short term interest rate at time t-1 under scenario s.

3.9. Basic Concepts of Stochastic Programming

We will now examine model for dynamic portfolio strategies defined and optimized in event trees. The models of this paragraph develop truly dynamic strategies that are optimal and satisfy the reasonable unexpected constraints. The balance of the portfolio is permitted as new information is available and the decisions of the portfolio do not depend on "fortune telling".

The stochastic programming is that mathematical programming tool that facilitates the optimization of dynamic strategies in event trees. We will introduce the basics of stochastic programming and then formulate a standard model for portfolio management.

To help in the understanding of stochastic programming problems, we must first look at a simple design problem under uncertainty such as the problem of the newsvendor. Then we will formulate two special cases of stochastic programs, the anticipative and the adaptive models. Finally we will combine these two in a most general formulation of the recourse model which is the most appropriate model for most financial applications.

3.9.1. The newsvendor problem

On a street corner a young entrepreneur sells newspaper which buys from a local distributor every morning. He sells them for a profit P_0^+ per unit, and left over newspapers in the end of the day are sold as waste paper, in which case there is a net loss P_0^- per unit. The demand for newspapers is a random variable $\tilde{\xi}$ which belongs to a probability space denoted by $\Xi = \{\tilde{\xi} \in \mathbb{R} | 0 \leq \tilde{\xi} \leq \infty\}$ and with probability function$P(\tilde{\xi})$. The problem is the choice of the optimal number of newspaper x to be purchased from the local distributor.

An approach for modeling this situation is to consider x the optimal, when it maximizes the expected profit. The profit of the function of x and the random value $\tilde{\xi}$. Suppose that $F(x; \tilde{\xi})$ is the profit function:

$$F(x;\ \tilde{\xi}) = \begin{cases} xP_0^+ & \text{if } x \leq \tilde{\xi} \\ P_0^+\ \tilde{\xi} - P_0^-\ (x - \tilde{\xi}) & if\ x > \tilde{\xi} \end{cases} \tag{3.13}$$

Where P_0^+ is the per unit profit, P_0^- is the loss per unit, x is the optimal number of newspapers, $\tilde{\xi}$ is a random variable that takes all values of the probability space that we defined above.

The expected value of the profit function is the integral with respect to the distribution function:

$$E[F(x;\ \tilde{\xi})] = \int_{\Xi} F(x;\ \tilde{\xi}) dP(\tilde{\xi}) = \int_0^X (\ P_0^+\ \tilde{\xi} - P_0^-\ (x - \tilde{\xi}) dP(\tilde{\xi}) + \int_x^\infty P_0^+ x\ dP(\tilde{\xi}) \tag{3.14}$$

And the mathematical model for the newsvendor problem is the following optimization problem with respect to x.

$$\text{Maximize } E[F(x;\ \tilde{\xi})]$$

$$\text{Subject to } x \geq 0$$

This is a simple example of a programming problem under uncertainty conditions. It is an adaptive model, since the decisions adjust as more information is available, i.e. as the newspapers are sold to customers who purchase during the day. The model has fixes recourse, which means that the observed reaction in demand is stable. That is, the number of newspapers sold for profit is determined solely by the number of customers. The same applies to the surplus generated at the end of the day, which is sold as waste paper at a loss. Other forms recourse action might be possible, such as the purchase of additional newspapers in higher cost later during the day or the return of newspapers before the end of the day at a price higher than that of the paper thrown away. This simple fixed recourse model does not allow such thoughts and also requires that all risk preferences are captured by the expected value of profit. At higher moments of the profit distribution function are ignored. The next section presents mathematical models for planning under uncertainty in more complicated situations.

3.10. Canonical stochastic programming problems

The next problem is the canonical formulation of stochastic programming:

$$\text{Minimize } \mathrm{E}\left[\mathrm{F}\left(\mathrm{x};\ \tilde{\xi}\right)\right]$$

$$\text{Subject to } \mathrm{E}\left[\mathrm{F}\left(\mathrm{x};\ \tilde{\xi}\right)\right]=0 \text{ for all } \mathrm{j}=1, 2,\ldots, \mathrm{m},$$

$$x \in X.$$

The following notations are used: $x \in \mathbb{R}^n$ is the vector of decision variables, $\tilde{\xi}$ is a random vector a space $\Xi \subset \mathbb{R}^N$ and $P = P(\tilde{\xi})$ is a probability distribution function on $\mathbb{R}^N$. Also $f_0: \mathbb{R}^n \times \Xi \to \mathbb{R} \cup \{\infty\}, f_j: \mathbb{R}^n \times \Xi \to \mathbb{R}, j = 1,2, \ldots \ldots, m$ and $X \in \mathbb{R}^n$ is a close set. Inequality constraints can be incorporated into this formulation with the use of slack variables.

The expectation functions:

$$\mathrm{E}\left[\mathrm{f_j}\left(\mathrm{x};\ \tilde{\xi}\right)\right] = \int_{\Xi} \mathrm{f_j}\left(\mathrm{x};\ \tilde{\xi}\right)\mathrm{dP}\left(\tilde{\xi}\right) \tag{3.15}$$

Are assumed finite for all j=1,2,…,m, unless the set $\{\tilde{\xi} | f_0(x; \tilde{\xi}) = +\infty\}$ has a non zero probability, in which case $E[f_0(x; \tilde{\xi})] = +\infty$. The feasible set we assume to be non empty.

The above model is a nonlinear programming problem whose constraints and objective function are represented by integrals. A big part of the theory of stochastic programming deals with the determination of the properties of these integrals functions and the development of appropriate approaches for their estimation. The calculation of the solutions for these non linear programs poses serious challenges, since the evaluation of the integrals can be an extremely difficult task, especially when the expectation functions are multidimensional.

There are even cases where the integrals are neither differentiable, nor convex, nor even continuous. A broad class of stochastic programming models however, can be formulated as large scale linear or non-linear programs in a specially structured constraints matrix.

3.10.1. Anticipative models.

We examine the situation where a decision x must be taken in an uncertain world where uncertainty is described by the random vector$\tilde{\xi}$. The decision cannot in any way depend on future observations, unlike the prudent planning that has to anticipate possible future realizations of the random vector. In anticipative models the feasibility is expressed in terms of probabilistic constraints. For example to be determined by a level of reliability a where 0≤a≤1 and constraints are expressed in the form:

$$P\left(\tilde{\xi} \middle| g_j\left(x;\tilde{\xi}\right) = 0, j = 1,2,\dots m\right) \geq a, \tag{3.16}$$

$$\text{Where } g_j : IR^n \times \Xi \to IR, j = 1,2,..m. \tag{3.17}$$

Where $\tilde{\xi}$ is a random vector that takes all values on a probability space and a is the level of reliability which means that is a constant that takes values between 0 and 1.

This constraint can be expressed in the form of the general model by defining f_i:

$$f_j\left(x;\tilde{\xi}\right) = \begin{cases} a - 1 \; if \; g_j\left(x;\tilde{\xi}\right) = 0 \\ \quad a \quad otherwise \end{cases} \tag{3.18}$$

The objective function can also be of a different type of reliability, as $P\left\{\tilde{\xi} \middle| g_0(x;\tilde{\xi} \leq \gamma\right\}$ where $g_0: \mathbb{R}^n \times \Xi \to \mathbb{R} \cup \{\infty\}$ and γ is a constant.

An anticipative model chooses a policy that leads to some desirable characteristics of the functions of the constraints and the objective functions in the realizations of the random vector. In the above example, it is desired that the probability of a constraint violation is less than predetermined threshold value a. The precise value of a depends on the application at hand, the cost of constraint violation and other similar considerations.

3.10.2. Adaptive models.

In an adaptive model observations related to uncertainty become available before taking a decision, so that optimization takes place in a learning environment. It is understood that observations provide only partial information about the random variables, otherwise the model will simply expect to observe the values of random variables and then decide by solving a deterministic mathematical program. In contrast to this situation we have other extreme where all observations are made after the decision x has been made, and the model becomes anticipative. Let A be the collection of all relevant information that could become available by making an observation. This A is a subfield of the σ-field of all possible events, generated from the support set Ξ of the random vector $\tilde{\xi}$. The decisions x depend on the events that could be observed, and x is termed A-adapted or A-measurable. Using the conditional expectation with respect to A, $E(\cdot\,|A)$, the adaptive stochastic program can be written as:

Model 3.3 Model of expected engaged value minimization.

Model of expected engaged value minimization

$$\text{Minimize } E\left(f_0\left(x(\tilde{\xi});\ \tilde{\xi}\right)\middle|A\right)$$

$$\text{Subject to } E\left(f_j\left(x(\tilde{\xi});\ \tilde{\xi}\right)\middle|A\right) = 0 \text{ for all } j = 1,2,\dots,m$$

$$x(\tilde{\xi}) \in X$$

Where $E\left(f_0\left(x(\tilde{\xi});\ \tilde{\xi}\right)\middle|A\right)$ is the objective function of expectations, x are the decisions, A is a subfield of the σ-field of all possible events and $\tilde{\xi}$ is a random vector that takes values into the probability space.

The representation $x{:}\,\Xi \to X$ is such that $x(\tilde{\xi})$ is A-measurable. This problem can be overcome by solving the following deterministic programs for each $\tilde{\xi}$:

Model 3.4: Deterministic model of expected engaged value minimization.

Deterministic model of expected engaged value minimization

$$\text{Minimize } E(f_0(x;\cdot)|A)\ (\tilde{\xi})$$

$$\text{Subject to } E(f_j(x;\cdot)|A)\ \left(\tilde{\xi}\right) = 0\ for\ all\ j = 1,2,\dots,m$$

$$x \in X$$

Where $E(f_0(x;\cdot)|A)\ (\tilde{\xi})$ is the objective function of expectations, x are the decisions, A is a subfield of the σ-field of all possible events and $\tilde{\xi}$ is a random vector that takes values into the probability space.

The two extreme cases (i.e. the complete information where A=Σ or no information) deserve special mention. The case of no information reduces the model to the form of the anticipative model. If there is full information (Model 3.3), this is known as the distribution model. The goal in this last case to characterize the distribution of the optimal objective function value. The exact values of the objective function and the optimal policy x are established after the observation of the outputs of the random vector $\tilde{\xi}$. The most interesting situations arise when it becomes available partial information, while some decisions are already taken.

3.10.3. Recourse models.

The recourse problem combines the anticipative and adaptive model in a common mathematical framework. The problem seeks a policy that not only expects future observations, but also takes into account that observations are made about uncertainty as time passes, and thus can adapt by taking recourse decisions. For example, a portfolio manager determines the composition of a portfolio by taking into account future movements in stock prices and that the portfolio will be balanced as prices change (adaptation). The two-stage version of this model is amenable to such formalities, as a large scale non-linear program with a special structure of the constraints matrix. For the formulation of the two stage stochastic program with recourse we need two vectors for decision variables to distinguish between the anticipative policy and the adaptive policy. The following notation is used:

The $x \in \mathbb{R}^{n_0}$ denotes the vector of first-stage decisions. These decisions are made before the random variables are observed and are anticipative. The $y(\tilde{\xi}) \in \mathbb{R}^{n_1}$ denotes the vector of second-stage decisions. These decisions are made after the random variables have been observed and are adaptive. They are constrained by decisions made at the first-stage, and depend on the realization of the random vector $\tilde{\xi}$.

We formulate the problem of the second stage in the following matter. Once a first-stage decision x has been made, some realization of the random vector can be observed. Let $q(y(\tilde{\xi}); \tilde{\xi})$ denote the cost function for the second stage decisions, and let $\{T(\tilde{\xi}), W(\tilde{\xi}), h(\tilde{\xi}) | \tilde{\xi} \in \Xi\}$ be the model parameters. Those parameters are functions of the random vector $\tilde{\xi}$ and are, therefore, random parameters. T is the technology matrix of dimension $n_1 \times m_0$. It contains the coefficients that convert the first-stage decision x into resources for the second-stage problem. The term "technology" refers to the fact that it is typically the changes in technology that determine the impact of today's decisions on the future decisions. W is the recourse matrix of dimension $n_1 \times m_1$ that imposes constraints on future decisions. h is the second-stage resource vector of dimension n_1. The second-stage problem seeks a policy $y(\tilde{\xi})$ that optimizes the cost of the second-stage decision for a given value of the first-stage decision x. We denote the optimal value of the second-stage problem by Q(x; $\tilde{\xi}$). This value depends on the random parameters and on the value of the first-stage variables x. Q(x; $\tilde{\xi}$) is the optimal value, for any given ω, of the following non linear program:

$$\text{Minimize } q(y(\tilde{\xi}); \tilde{\xi})$$

$$\text{Subject to } W(\tilde{\xi})y(\tilde{\xi}) = h(\tilde{\xi}) - T(\tilde{\xi})x$$

$$y(\tilde{\xi}) \geq 0$$

Where $y(\tilde{\xi})$ is the recourse decision of the second stage, x are the decisions, $\tilde{\xi}$ is a random vector that takes values into the probability space.

If this second-stage problem is infeasible then we set Q(x; $\tilde{\xi}$) =+∞. The upper model is an adaptation model in which $y(\tilde{\xi})$ is the recourse decision and Q(x; $\tilde{\xi}$) is the recourse cost function. The two-stage stochastic program with recourse is an optimization problem in the first-stage variables x, which optimizes the sum of the cost of the first-stage decisions, f (x), and the expected cost of the second-stage decisions. It is written as follows:

Model 3.5: Two stages stochastic programming model

Two stages stochastic programming model

$$\text{Minimize } f(x) + E[Min\{q(y(\tilde{\xi});\tilde{\xi})|T(\tilde{\xi})x + W(\tilde{\xi})y(\tilde{\xi}) = h(\tilde{\xi}), y(\tilde{\xi}) \geq 0\}]$$

Subject to Ax=b

x≥0

Where $y(\tilde{\xi})$ is the decision vector of the second stage, x are the decisions, $\tilde{\xi}$ is a random vector that takes values at a probability space.

Let us denote $K_1 = \{x \in \mathbb{R}_+^{n_0} | Ax = b\}$ symbolizes the feasible set for the first stage problem. Also $K_2 = \{x \in \mathbb{R}_+^{n_0} | E[Q(x;\widetilde{\xi)}]\} \leq +\infty$ denote the set of induced constraints. This the decision set of the first stage x for which the second stage problem is feasible. The previous problem has a complete recourse if$K_2 = \mathbb{R}^{n_0}$, that is if the second-stage problem is feasible for every value of x. The problem is relatively complete recourse if $K_1 \subseteq K_2$, that is a second stage problem feasible for every value of the first-stage variables that satisfies the first-stage constraints. Simple recourse refers to the case of the resource matrix $W(\tilde{\xi}) = I$ and the recourse constraints take the simple form $I_{y+}(\tilde{\xi}) - I_{y+}(\tilde{\xi}) = h(\tilde{\xi}) - T(\tilde{\xi})x$ where I is the identity matrix and the recourse vector $y(\tilde{\xi})$ is written as $y(\tilde{\xi}) = y + (\tilde{\xi}) - y - (\tilde{\xi})$, with $y + (\tilde{\xi}) \geq 0, y - (\tilde{\xi}) \geq 0$ almost surely.

3.11. Deterministic equivalent formulation

We take now the case that the random vector $\tilde{\xi}$ has a discrete and finite distribution and$\Xi = \{\xi^1, \xi^2, \ldots, \xi^N\}$. The elements ξ^l of Ξ are scenarios with the index l from the scenario sample spaceΩ. We denote with p^l the probability of realization of the l-th scenarioξ^l. That is for every$l\epsilon\Omega$.

$$p^l = Prob(\tilde{\xi} = \xi^l) =$$
$$= Prob\left\{\left(q(y;\tilde{\xi}), W(\tilde{\xi}), h(\tilde{\xi}), T(\tilde{\xi})\right) = (q(y;\xi^l), W(\xi^l), h(\xi^l), T(\xi^l))\right\}$$
$$\text{Where } p^l > 0 \; for \; all \; l \in \Omega \; and \; \sum_{l\in\Omega} p^l = 1 \quad (3.19)$$

The expected value if the optimization problem if the second stage is expressed as follows:

$$E\big(Q(x;\tilde{\xi})\big] = \sum_{l\in\Omega} p^l Q(x;\xi^l) \quad (3.20)$$

For every realization of the random vector $\xi^l, l \in \Omega$ a different second-stage decision is made $y(\xi^l)$, which is denoted for by y^l. The second-stage problems can then be written as:

$$\text{Minimize } \; q(y^l;\, \xi^l)$$
$$\text{Subject to } W(\xi^l)y^l = \; h(\xi^l) - T(\xi^l)x \quad (3.21)$$
$$y^l \geq 0$$

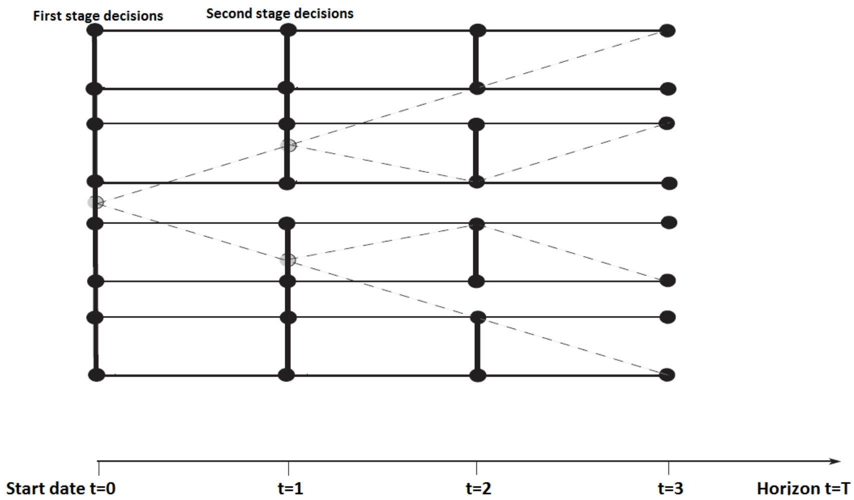
If we combine the equations 3.20 and 3.21 we can rephrase the stochastic non linear model 3.5 as a large scale deterministic equivalent nonlinear program.

3.12. Spit variable formulation.

Requiring non-predictability, that a decision cannot be based on information arriving at later intervals, leads to the model above in which the first stage decision x is common for all subsequent scenarios. An equivalent presentation, which is intuitively more appealing, is to allow different first stage decisions x^l for each scenario $l \in \Omega$, but thus we are forcing these variables to be equal to each other with additional explicit restrictions.

We can have a better understanding in this new version under an event tree. The tree is divided into linear scenarios whenever there is branching. This creates a multiple split-states (separated financial statements), but the new economy states for the same period and the same state of the economy from the event tree must be the same. Figure 3.4 illustrates the linear scenario structure that is derived from the decompositions of the tree in Figure 3.3. The split-states should coincide and be connected by thick solid lines.

Figure 3.4.: Splitting an event tree into linear scenarios: dotted lines show the event tree, solid lines show the linear scenarios and thick lines are used to denote the presence of non-anticipativity constraints.



3.13. Stochastic programming for Dynamic strategies.

We now examine the optimization of dynamic portfolio strategies in event trees. In each trading date the manager must evaluate the market conditions such as prices and interest rates, which are predominant in the currents state of the economy. The administrator also must evaluate possible fluctuations in interest rates, prices and cash in possible states of the economy at the next negotiating period. This means that information on an upcoming state of the economy should be evaluated.

This information is incorporated into a series of transactions for the purchase or sale of securities and short-term borrowing or lending. Over the next trading date, the portfolio manager holds a "tough" portfolio and is confronted with a new set of likely future movements. The administrator has to incorporate the information so that the transactions can be executed. The model defines a sequence of investment decisions in discrete trading time. The decisions are taken at the beginning of each period. The portfolio manager starts with a portfolio and a series of scenarios for future states of the economy, which are incorporated into an investment decision. The composition of this portfolio depends on th transactions in the previous decision point in a scenario that was implemented in the meantime. Another set of investment decisions incorporates both the current status of the portfolio and new information about future scenarios. We will now present a model of stochastic programming for dynamic strategies.

3.14. Model presentation

There are two basic constraints in most stochastic programming models for portfolio optimization. The first one considers the cash flows for safe assets, cash, and the other one is an equation of the inventory balance sheet for each security or asset in all negotiating dates and for all states of the economy. The Figure 3.5 presents the cash flow and the Figure 3.6 the inventory of assets categories.

We also formulate the characteristics of the model for $t = 0$ and for future bargaining dates$0 \leq t \leq T$. For $t = 0$ we have the first stage problem. The variables for future negotiation dates are t-stage variables used for modeling the recourse model.

3.14.1. First-stage constraints

At the first stage all the variables are known with certainty and we also know the composition of the portfolio. For every security or asset category $i \epsilon \cup$ in the portfolio we have an inventory balance constraint.

$$z_{0i}^{0} = b_{0i} + x_{0i}^{0} - y_{0i}^{0} \tag{3.22}$$

Where z_{0i}^{0} is the initial inventory, $b_0 = (b_{01}, \dots \dots \dots b_{0n})$ the initial stock for every asset category, x_{0i}^{0} the initial purchases and y_{0i}^{0} the initial sales as shown below in Figure 3.6.

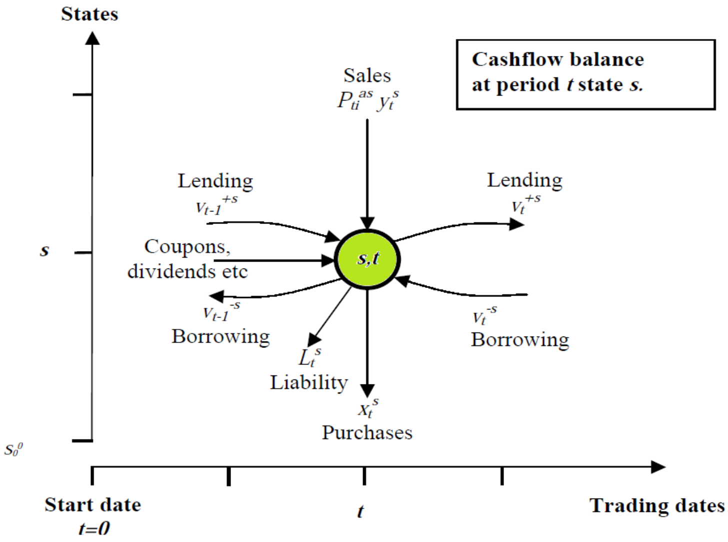
The equation of the cash flows balance sheet determines that the initial endowment in safe assets, and income from the sale of part of the existing portfolio, are equal to the amount invested for the purchase of new securities plus obligations payment and of the amount invested in the asset without risk, i.e.

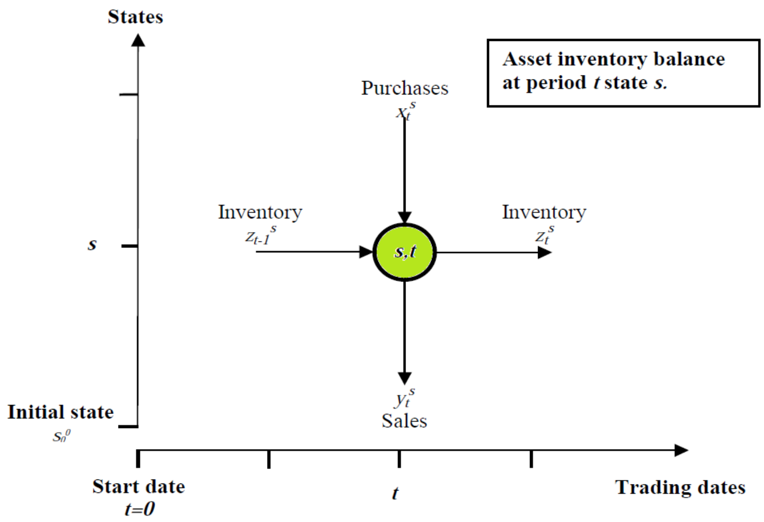
$$\sum_{i=1}^{n} P_{0i}^{b0} y_{0i}^{0} + v_0 + v_0^{-0} = \sum_{i=1}^{n} P_{0i}^{a0} x_{0i}^{0} + v_0^{+0} + L_0^0 \tag{3.23}$$

3.14.2. Time staged constraints

The decisions taken in future trading dates $t = 1,2, \dots, T$ are conditioned on the state of the economy $s \in \sum_t$. Therefore in every state of the economy we have a number of constraints for each state of the economy. These decisions depend on the investing decisions taken in previous trading dates $t - 1$ at predecessors^-.

Figure 3.5 & 3.6.: Flow of cash and asset inventory in a stochastic programming model for dynamic portfolio strategies at trading date t in state s.





Asset inventory balance sheet equations limit the amount of each security that is sold or remains in the portfolio to be equal to the outstanding amount of face value carried over the previous trading date, as well as any amount purchased during the current trading date. There is a constraint for every security $i \epsilon U$ and every state of the economy $s \in \sum_t$.

$$z_{ti}^{s} = a_{(t-1)i}^{s} z_{(t-1)i}^{s^-} + x_{ti}^{s} - y_{ti}^{s} \qquad (3.23)$$

Where a denotes the depreciation factors, z_{ti}^s balance of inventory of fixed assets, x_{ti}^s the purchases and y_{ti}^s the sales.

The balance sheet of cash requires the amount invested in the acquisition of new titles and safe assets to be equal to the income generated by the existing portfolio during the holding period, plus any cash flows from sales and cash reinvested in the previous period to the predecessor s^{-} minus liabilities payments, see in Figure 3.5. There is a constraint for each state of the economy $s \in \sum_t$.

$$\sum_{i=1}^{n} F_{(t-1)i}^{s} z_{(t-1)i}^{s^{-}} + \sum_{i=1}^{n} P_{ti}^{bs} y_{ti}^{s} + \left(1 + r_{f(t-1)}^{s}\right) v_{t-1}^{+s^{-}} = L_t^s + \sum_{i=1}^{n} P_{ti}^{as} x_{ti}^{s} + v_t^{+s} \quad (3.24)$$

Where $F_{(t-1)i}^{s}$ are the cash flows invested in period$t-1$, $z_{(t-1)i}^{s^{-}}$ the inventory balance sheet in period$t-1$, P_{ti}^{bs} is the values of sales, x_{ti}^{s} the purchases and y_{ti}^{s} the sales, P_{ti}^{as} the value of purchases, L_t^s the value of liabilities, $r_{f(t-1)}^{s}$ is the short-term risk free interest rate of the previous period and v_t^{+s} is the surplus of the current period.

This constraint takes into account the investment in safe assets, in the previous period and in the predecessor state, but not borrowing. But the borrowing can be incorporated in this equation by introducing the variable v_t^{-s}. The loan will contribute to the cash inflow side in the left part of the equation, but borrowing from previous periods must be paid back, in upcoming periods. This will increase the cash outflows on the right side of the above equation. The equation of balance cash flows by borrowing and reinvest in every state of the economy $s \in \sum_t$ is written as follows:

$$\sum_{i=1}^{n} F_{(t-1)i}^{s} z_{(t-1)i}^{s^{-}} + \sum_{i=1}^{n} P_{ti}^{bs} y_{ti}^{s} + \left(1 + r_{f(t-1)}^{s}\right) v_{t-1}^{+s^{-}} + v_t^{-s} = L_t^s + \sum_{i=1}^{n} P_{ti}^{as} x_{ti}^{s} + v_t^{+s} + \left(1 + r_{f(t-1)}^{s} + \delta\right) v_{t-1}^{s^{-}} \quad (3.25)$$

Where δ is the spread between short-term borrowing and lending interest rate and v_t^{-s} is the amount of lending, $F_{(t-1)i}^{s}$ are the cash flows invested in period$t-1$, $z_{(t-1)i}^{s^{-}}$ the inventory balance sheet in period$t-1$, P_{ti}^{bs} is the values of sales, x_{ti}^{s} the purchases and y_{ti}^{s} the sales, P_{ti}^{as} the value of purchases, L_t^s the value of liabilities, $r_{f(t-1)}^{s}$ is the short-term risk free interest rate of the previous period and v_t^{+s} is the surplus of the current period.

3.14.3. End of horizon constraints.

At the end of the planning horizon we evaluate the final wealth of the portfolio. This will depend on investments in various asset classes including cash and the states of the economy. It is given by:

$$W_T^s = v_T^{+s} + \sum_{i=1}^{n} P_{Ti}^{bs} z_{ti}^{s} \quad (3.26)$$

Where W_T^s is the wealth of the portfolio at the end of the horizon v_t^{+s} is the surplus at the end of the horizon, P_{Ti}^{bs} is the price of sales at the end of the horizon and z_{ti}^s are the sales.

3.14.3. The Objective function.

To incorporate risk aversion in dynamic portfolio strategy we introduce a utility function for the final wealth. The objective function of the portfolio optimization model maximizes the expected utility of final wealth.

$$Maximize \sum_{s \in \sum_T} p^s U(W_T^s) \tag{3.27}$$

Where p^s is the probability connected with the state of the economy s at $\sum_t$, W_T^s is the final wealth given by the equation 3.26 and U symbolizes the utility function.

Generally this is not the only utility function that we can use. There are other options that might be more suitable for some applications. Furthermore the creation of an objective function for investors for long time horizons is a difficult part. First we need to evaluate the short –term versus long- term trading. Secondly uncertainty for extended periods of time complicates the decision-making process. The choice of an objective function of an objective function that reconciles the preference theories of investors and utility functions is the most important step in the modeling process.

3.15. Stochastic programming method and comparison with other methods.

Stochastic programming belongs in the category of multi-period stochastic models for risk management. We show in previous chapters many risk measures one of them was Value at Risk (VaR) and also a risk measure that belongs in the coherent risk measure category known as conditional value at risk (CVar) used by many managers for the efficient selection of securities in a portfolio. We show that we could use the mean-variance portfolio optimization with the use of mean-variance analysis as a normative model and then we introduced the Arbitrage pricing theory as the theoretical foundation for multi-factor models. Furthermore there is scenario optimization where methods like Monte Carlo simulation could be used to generate scenarios of security prices and returns. Due to nonlinear relationships between prices and risk factors the distributions of returns generated by the simulations can be highly asymmetric, skewed and have long tails and because of nonlinearities and asymmetries we cannot use the mean-variance optimization models by examining variance as a risk measure, but we could use downside-risk measures as a substitute of variance to solve some of these problems.

Also there are methods for the generation of scenarios for asset classes such as bootstrapping of historical data and as far as statistical modeling techniques are concerned there is the Value-at-Risk approach and time series analysis. There are discrete time multi period models and continuous time models. Both of the categories when solved with dynamic programming can provide significant qualitative insights about fundamental issues in investments and risk management. However their practical use as a tool for decision making is limited by the many simplifying assumptions that are needed to derive the solutions in reasonable amount of time. In stochastic programming many issues such as transaction cost, taxes and other variables can be handled simultaneously but the computational effort that is needed increases as the number of decision stages increases, such a method is not always solvable there are too many details and two many variables that could confuse the decision maker instead of inform him. Most of the time capturing the first few opportunities accurately is good enough so as to make a decision at the moment. So stochastic programming may have its advantages but it has also shortcomings and in that case other simpler methods that need less time to construct and evaluate the results are better when we need to make a specific decision and manage our portfolio in practice. Other methods for calculating Var so as to optimize our portfolio are more utilitarian, a popular one is the historical simulation approach and also there is the model-building approach. Moreover volatility is monitored by two very popular models that recognize it as non constant and keep track of its variation through time. These models are the exponentially weighted moving average (EWMA) and the generalized autoregressive conditional heteroscedasticity (GARCH). In practice the second one GARCH because it incorporates the mean [7]reversion while the EWMA does not, theoretically is more appealing.

Chapter 4th

OVERVIEW OF RESEARCH METHODOLOGY

4.1. Introduction

We presented in the previous chapter a recent technique to optimize a portfolio that is rather difficult computationally and not always needed when we have to take instant investment decisions and now we are going to present some techniques to determine the best portfolio composition of shares based in the

[7] Variance rates tend to be pulled back to a long-run average level, this is known as mean reversion.

mean-variance model, which we introduced in the first chapter of the thesis where we introduced the Markowitz theory that was also evolved by Tobin, because in the next chapter we are going to use them to construct the empirical part of the essay. This analysis can easily be extended in portfolio of other assets, not only shares.

We will show how the mean and the variance of a portfolio is calculated. Insert the notions of covariance and the correlation coefficient. We are also going to see how we can define the composition and calculate the optimal weights of a portfolio and what is the efficient frontier and the tangent portfolio mainly in the case of a risk free asset included in a portfolio, but also without it.

4.2. Expected return and variance of a portfolio without and with a risk free asset.

If we have a portfolio P that consists by two shares $i = \{1,2\}$ and symbolize their returns with r_i , their expected mean value is represented by $\overline{r_i} \equiv E(r_i)$ and their variance by ${\sigma_i}^2 = Var(r_i) = E[r_i - E(r_i)]^2$. In practice we calculate the expected value and the variance using historical as we are going to do in the next chapter. The return of a share is its performance rate:

$$r_i = P_{i,1} - P_{i,0}/P_{i,0} \quad (4.1)$$

Where $P_{i,1}$ and $P_{i,0}$ are the prices of a share i in the start and in the end of a period.

If we define the weights of its share in the portfolio w_1 and w_2 respectively and $w_1 + w_2 = 1$ then the return and the variance of the portfolio is calculated as follows:

$$\overline{r_p} = E(r_p) = E[w_1 r_1 + w_2 r_2] = w_1\overline{r_1} + w_2\overline{r_2} = w\overline{r_1} + (1-w)\overline{r_2} \quad (4.2)$$

$${\sigma_p}^2 = Var(r_p) = {w_1}^2{\sigma_1}^2 + {w_2}^2{\sigma_2}^2 + 2w_1 w_2 \rho_{12}\sigma_1\sigma_2 \quad (4.3)$$

Where $\overline{r_p}$ is the portfolio return, ${\sigma_p}^2$ is the variance of the portfolio and ρ_{12} is the correlation coefficient where $\rho_{12} = Cov(r_1;r_2)/\sqrt{{\sigma_1}^2\sqrt{{\sigma_2}^2}}$ and $Cov(r_1;r_2)$ is the covariance of the returns of the two shares.

By the relationships of the returns and variances we know that the return of the portfolio is between the returns of the assets that the portfolio consists. That means that $\overline{r_1} < \overline{r_p} < \overline{r_2}$ and also that the variance of the portfolio ${\sigma_p}^2$ could be smaller than the other two variances, ${\sigma_p}^2 < {\sigma_1}^2 or\ {\sigma_p}^2 < {\sigma_2}^2$. That happens because investing in a portfolio of assets rather than an individual asset, as we have already said in other chapters, is better due to the benefits of diversification.

Moreover less investing risk mean smaller variance of the portfolio, which depends as we saw in the equation 4.3 from the covariance of the assets or else from the correlation coefficient. The correlation coefficient has no unit of measurement and is a way to measure the diversification of the portfolio.

For the correlation coefficient, which is known by Spearman [1904] also as Spearman correlation coefficient, applies that $-1 < \rho < 1$. When the correlation coefficient is negative means that there is a negative correlation between the two variables, in this case assets, and when one increases the other decreases, that correspond to a decreasing monotonic trend among the two variables. On the other hand when the correlation coefficient is positive, that corresponds to a monotonous ascending trend among the two variables, when one increases the other increases too. When the correlation coefficient of two assets is $\rho_{12} = 1$ means that the returns of the assets are linearly correlated and move to the same direction, when $\rho_{12} = -1$ means that the moves of the returns are exactly the opposite. Likewise when the correlation coefficient equals zero $\rho_{12} = 0$ the returns of the two assets are linearly uncorrelated.

Also now that we defined the return and the variance of the portfolio we have to observe that we can calculate the Sharpe index which we presented in the second chapter and is ratio between return and variance. The bigger the Shape index the better the portfolio that we constructed compared with others with smaller Sharpe index.

Now when we also have a risk free asset in the portfolio with certain returns denoted by r_f and with portfolio weight w_0, where $w_0+w_1 + w_2 = 1$ then the portfolio's return and variance is calculated as follows:

$$\overline{r_p} = E(r_p) = w_0 r_f + w_1\overline{r_1} + w_2\overline{r_2} \quad (4.4)$$

In the case of the risk free asset the portfolio's variance is the same as in equation 4.3, because the risk free asset has certain returns and does not pose risk so the variance of the portfolio is not written as a function ofw_0. Furthermore w_0 is going to be defined as a residual $w_0 = 1 - w_1 - w_2$ after defining the optimal values of w_1 andw_2. Then the equation 4.2 will be written as follows:

$$\overline{r_p} = E(r_p) = (1 - w_1 - w_2)r_f + w_1\overline{r_1} + w_2\overline{r_2} = r_f + w_1(\overline{r_1} - r_f) + w_2(\overline{r_2} - r_f) \quad (4.5)$$

Where $(\overline{r_1} - r_f)$ and $(\overline{r_2} - r_f)$ are the excess returns. The case of the risk free asset is what concerns because it is what we are going to examine in the empirical part.

4.3. Expected return and variance in the case of N assets.

We are going to generalize the equations of the two asset portfolio and show some calculations for a portfolio of N assets. We have the case of a portfolio consisting by N assets with expected returns $E(R_1), E(R_2), \dots, E(R_n)$, standard deviations $\sigma_1, \sigma_2, \dots., \sigma_n$ and participation percentages (weights) $w_1 + w_2 + \cdots w_n = 1$.

The equation of the expected return and standard deviation are the following:

$$E(R_p) = \sum_{i=1}^{n} w_i E(R_i) \tag{4.6}$$

$$\sigma_p = \sqrt{\sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}} \quad i \neq j \tag{4.7}$$

Furthermore the above equation could be written and calculated by the use of vectors and tables. The expected return and the standard deviation are expressed in a simpler way as follows:

$$E(R_p) = w^T E(R) = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}^T \begin{bmatrix} E(R_1) \\ E(R_2) \\ \vdots \\ E(R_n) \end{bmatrix} \tag{4.8}$$

Where $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}^T$ is the transpose table w of the weights of the n assets with size $n \times 1$

and $\begin{bmatrix} E(R_1) \\ E(R_2) \\ \vdots \\ E(R_n) \end{bmatrix}$ is the table of the expected returns of the n assets with size $n \times 1$.

$$\sigma_p = \sqrt{w^T V w} = \sqrt{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}^T \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}} \tag{4.9}$$

Where $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}^T$ is the transpose table w of the weights of the n assets with size $n \times 1$

and $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$ is the table w of the weights of the n assets with size $n \times 1$ and $\begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{bmatrix}$ is the table V with size $n \times n$ that includes the variances of the assets on the diagonal, while all the other elements are the covariances of returns of the assets and the matrix is symmetrical

4.4. Efficient portfolio.

As we said in the first chapter of the thesis the diversification is the base line of the mean-variance model of Markowitz and without it we cannot conclude to an efficient portfolio. The diversification consists in forming a portfolio of as many as possible and properly selected assets in order to reduce a significant percentage of the overall performance of the portfolio. A key component of diversification is the selection of assets whose returns are uncorrelated with each other, the correlation coefficient tend to zero. The diversification is a shield against non-systematic risk.

Efficient portfolio is the one that for a certain level of risk presents the maximum level of return and conversely for a certain level of return presents the minimum level of risk. The efficient portfolio as an optimal combination between risk and return is not unique, it depends on which of the two variables we want to optimize by keeping the other one constant and produce as many efficient portfolios as we want. The efficient portfolios form a geometrical space called efficient frontier, we will present extensively later in the thesis. The portfolio of the efficient frontier that has the smallest risk is called minimum variance portfolio and the portfolio of the efficient frontier that has the largest return is called maximum return portfolio. Those two portfolios help us define the efficient frontier.

The best portfolio of all the efficient portfolios that an investor chooses is called optimal portfolio and depends on the preferences in the risk-return trade-off. The preferences can be found by the utility function of the investor.

4.5. The determination of optimal weights.

The optimal weights of a portfolio P consisting for example by two shares as we said earlier could be mathematically calculated by solving the problem of the minimization of the variance ${\sigma_p}^2$ for a certain value μ of the return $\overline{r_p}$.

$$min_{w_1,w_2}{\sigma_p}^2 = {w_1}^2{\sigma_1}^2 + {w_2}^2{\sigma_2}^2 + 2w_1w_2\sigma_{12} \tag{4.10}$$

This problem is solved under two constraints:

$$\overline{r_p} = w_1\overline{r_1} + w_2\overline{r_2} = \mu \tag{4.11}$$

$$w_1 + w_2 = 1 \tag{4.12}$$

The relationship 4.7 sets the return equal to μ and the relationship 4.8 is the income constraint. To solve this problem we use the Lagrange method. We denote by λ_1 and λ_2 the Lagrange multipliers. Then we have the following equation:

$$min_{w_1,w_2} L = \frac{1}{2}[{w_1}^2{\sigma_1}^2 + {w_2}^2{\sigma_2}^2 + 2w_1 w_2 \sigma_{12}] + \lambda_1[\mu - w_1\overline{r_1} - w_2\overline{r_2}] + \lambda_2[1 - w_1 - w_2] \quad (4.13)$$

To minimize the problem given in the above equation some first order conditions that have to apply:

$$\frac{\partial L}{\partial w_1} = w_1{\sigma_1}^2 + w_2\sigma_{12} - \lambda_1\overline{r_1} - \lambda_2 = 0 \quad (4.14)$$

$$\frac{\partial L}{\partial w_2} = w_2{\sigma_1}^2 + w_1\sigma_{12} - \lambda_1\overline{r_2} - \lambda_2 = 0 \quad (4.15)$$

$$\frac{\partial L}{\partial \lambda_1} = [\mu - w_1\overline{r_1} - w_2\overline{r_2}] = 0 \quad (4.16)$$

$$\frac{\partial L}{\partial \lambda_2} = [1 - w_1 - w_2] = 0 \quad (4.17)$$

We also use the variance-covariance matrix represented by Σ and is equal to: $\Sigma = \begin{bmatrix} {\sigma_1}^2 & \sigma_{12} \\ \sigma_{12} & {\sigma_2}^2 \end{bmatrix}$. The vector of the returns is represented by $\bar{r} = \begin{bmatrix} \overline{r_1} \\ \overline{r_2} \end{bmatrix}$ and by solving with this method we first come up with the following equation for the weights:

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda_1\, \Sigma^{-1}\bar{r} + \lambda_2\, \Sigma^{-1}\bar{r}1 \quad (4.18)$$

Where $1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

By solving we symbolize by $a = \bar{r}'\Sigma^{-1}\bar{r}$, $b = \bar{r}'\Sigma^{-1}1$, $c = 1'\Sigma^{-1}1$ and they are vectors and we conclude that the Lagrange multipliers are equal to:

$$\lambda_1 = \frac{\mu c - b}{ac - b^2} \quad (4.19)$$

$$\lambda_2 = \frac{a - \mu b}{ac - b^2} \quad (4.20)$$

And finally the optimal weights are denoted be w^* and are found by setting the equations 4.10 and 4.11 to the equation 4.9 and solving it.

In the case of the risk free asset if we want to find the optimal weights we solve with the same way the equation 4.6 the differences are that $\overline{r_p}$ is now given by the equation 4.5 and the second constraint 4.8 does not apply anymore, because now there is to possibility of lending in the market. We have only one constraint so there is only one Lagrange multiplier λ and in this case is given by the following equation:

$$\lambda = \frac{\mu - r_f}{g} \qquad (4.21)$$

Where $g = \bar{r}^{*'} \Sigma^{-1} \bar{r}^{*}$ and $\bar{r}^{*} = \bar{r} - 1 r_f$. In this case, of the risk free asset, the weights are denoted by w^* and given by:

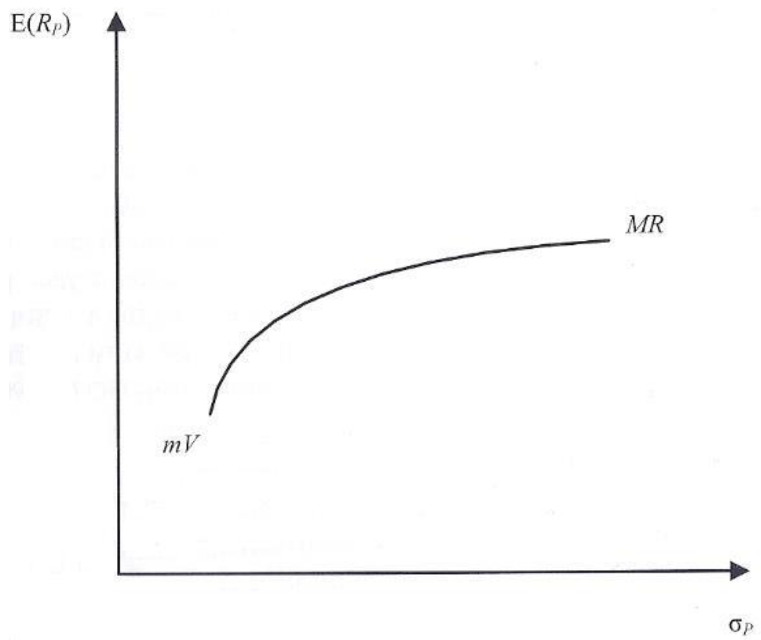
$$w^* = \left(\frac{\mu - r_f}{g}\right) \Sigma^{-1} \bar{r}^{*} \qquad (4.22)$$

By the same way we solve for N assets.

4.6. The efficient frontier and the tangent portfolio.

The efficient frontier as we said earlier is a subset of portfolios, which are preferred by all investors that are risk averse and prefer greater expected returns. Those portfolios are called efficient and form the efficient frontier, whose graph differs depending on the assumptions made to the existence of short selling and riskless borrowing or lending.

Figure 4.1: Efficient frontier no short-selling.



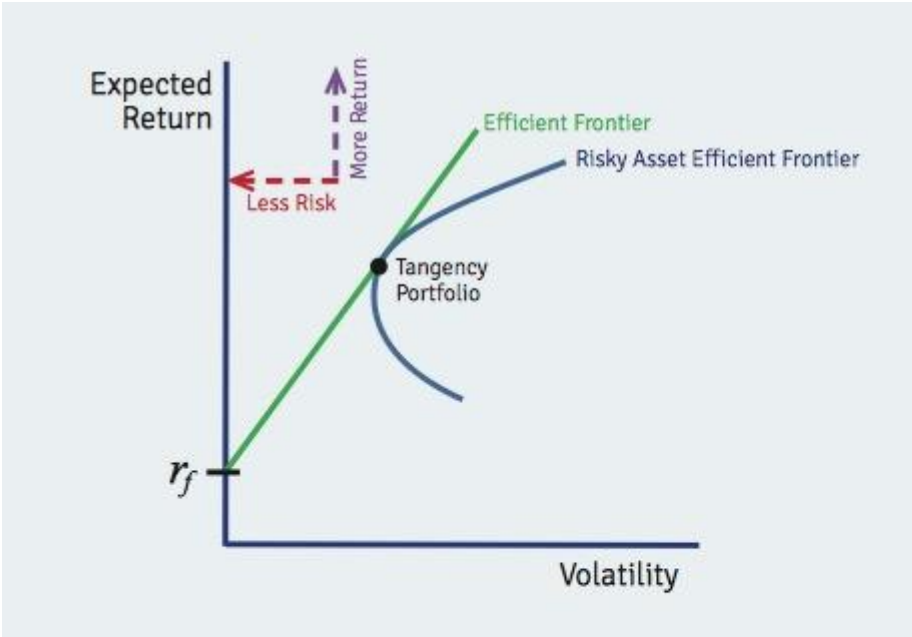
In Figure 4.1 we can see the relationship between expected return and standard deviation. We can see that all the portfolios that are located at the curve have the ability, for a certain value of portfolio return $\overline{r_p}$ to provide the smallest possible standard deviation σ_P, because of this ability this curve is called the efficient portfolio frontier denoted by EPF.

The portfolios that are at that curve have the largest possible ratio of expected return and standard deviation, which means the largest Sharpe index between all the feasible portfolios that are exactly at the curve and at the interior of the curve. These portfolios should be the investor's best possible choices.

All the portfolios that are under and at the interior of the curve should not be considered as efficient, because there are other portfolios at the curve that have for the same risk-standard deviation larger expected return. In contrast to the portfolios that are at the interior of the curve all the portfolios that are above and at the exterior of the curve are considered non feasible portfolios.

In Figure 4.1 we can see that the efficient frontier under the assumption of no short shelling is a concave function that extends from the minimum variance portfolio to the maximum return portfolio. In the case of the existence of short selling the efficient frontier extends from the minimum variance portfolio to infinity. In both cases the efficient frontier is a concave function and does not include any convex parts.

Figure 4.2: Expected return-standard deviation relationship.



Source: Investopedia

As we can see in the Figure 4.2 the straight line is the efficient frontier when there is a risky free asset in the portfolio. Also there is a point, when a risk free asset is included at the portfolio, where $w_0 = 0$ *and* $w_1 + w_2 = 1$. This portfolio at this point that includes only the two shares satisfies the constraint $w_1 + w_2 = 1$. This portfolio is at the efficient frontier when a risk free asset exists at the portfolio and also at the curve of the risky efficient frontier when there are only the shares in the portfolio. It is a common point of the two frontiers and is called a tangency point and that is why the portfolio that corresponds at this point is also called tangent portfolio and is denoted by T.

4.7. Calculation of the efficient frontier.

We will present in brief the way of calculating the efficient frontier in portfolios of N assets. We will calculate it first under the assumptions of no short selling and no riskless lending and borrowing. The efficient set of portfolios is defined from all the combinations that minimize risk for every level of expected return. To calculate any point on the efficient frontier we minimize the risk of the portfolio under the constraints of a constant level of expected return, of investing the whole capital at the portfolio and under the assumptions, all these are expressed as follows:

Minimize: $\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$ with $i \neq j$

Subject to the constraints: $\sum_{i=1}^{n} w_i E(R_i) = E(R_P) = \overline{R_P}$ (1)

$$\sum_{i=1}^{n} w_i = 1 \qquad (2)$$

$$w_i \geq 0, i = 1, \ldots\ldots, 8 \qquad (3)$$

The calculation of the efficient frontier is an optimization problem, but also a quadratic programming problem, because the constraints are linear but the objective function consists by quadratic terms and cross-product terms. The above could be written with the use of vectors and tables that we presented earlier.

The points of the minimized function of risk that satisfy the constraints are points-portfolios of the efficient frontier. As we said earlier the efficient frontier due is a concave function that extends from the minimum risk portfolio until the maximum return portfolio, due to the assumptions of no short selling and no risk lending or borrowing.

When there is a risk-free asset in the portfolio and under the assumptions of no short selling and the existence of riskless lending and borrowing the variance of the portfolio is the same as before and the expected return is defined by the equation that we presented earlier. In this case the efficient frontier is a straight line that starts from the point (0, R_F) and passes from the tangent point that we talked about it earlier and this part of the efficient frontier is preferable off all types of investors. To calculate it in the case of no short selling we have to find the straight line with the maximum slope. The slope of the line is the ratio of excess return of the portfolio, which is the difference between the expected return of the portfolio and the risk free rate, to the standard deviation of the portfolio. The mathematical representation of the quadratic programming problem to determine the efficient frontier is expressed as follows:

Maximize: $\frac{\overline{R_P}-R_F}{\sigma_P}$

Subject to the constraints: $\sum_{i=1}^{n} w_i = 1$ (1)

$w_i \geq 0, \quad \forall i$ (2)

Chapter5th

EMPIRICAL FINDINGS

5.1. Introduction

In this chapter it will be presented an application of the mean-variance model. The main purpose of the application of the Harry Markowitz and James Tobin theory is to find the efficient frontier in two cases of portfolios. The first case is a portfolio containing the general indexes of several countries in the Euro zone and the second a portfolio containing those indexes and as a risk free asset we use the 10-year German benchmark bond.

We examine first the risky assets portfolio containing eight general stock indexes by using the historical monthly prices in a five year period, from 01/01/2010 until 31/12/2014.Our purpose is to calculate the expected return and the risk of each index. Then based on the mean-variance model we calculated the expected returns, variances, standard deviations of the indexes as well as the variance-covariance and the correlation matrix. Afterwards we defined the non-linear programming optimization problem and its solution algorithm to find the optimal weights of the risky assets portfolio and presented in a graph the efficient frontier of a risky asset portfolio.

After calculating the efficient frontier of the risky assets portfolio we put into our portfolio the 10-year German benchmark bond and we study the new portfolio, so as to calculate after the input of the government bond, the optimal weights and the new efficient frontier of the eight risky assets and a risk free. We define this optimization problem such as in the first case, with an objective function and some constraints. In both cases is used the computational package of Microsoft Excel and the Solver Add-In to calculate the statistical measures and to solve quadratic programming problems. Moreover we also calculate the tangent portfolio and in the end we compare the efficient frontiers of the portfolios to ASE or else ATHEX stock index, which is the general stock index of Greece.

5.2. Presentation of the data.

To extract the data we used Bloomberg Terminal.We used historical monthly prices of stock indexes from 01/01/2010 until 31/12/2014. The indexes that we used are from countries in the Euro zone and as we can see the data are collected from a period during the economic crisis. We used the stock index of Italy (FTSE MIB), of Spain (IBEX-35), of Netherlands (AEX), of France (CAC-40), of Austria (ATX), of Belgium (BEL-20), of Germany (DAX), of Finland (OMXH-25) and of Greece (ASE). In the following table we present the historical monthly prices of the indexes.

Table 5.1: Historical prices of the indexes.

ASE	FTSE MIB	IBEX-35	AEX	CAC-40	ATX	BEL-20	DAX	OMXH-25
2048,32	21896,3	10947,7	327,9	3739,46	2493,53	2505,2	5608,79	2057,18
1913,16	21068,3	10333,6	317,74	3708,8	2438,95	2514,87	5598,46	2071,79
2067,49	22848	10871,3	344,22	3974,01	2634	2648,46	6153,55	2246,04
1869,99	21562,5	10492,2	345,91	3816,99	2650,32	2560,99	6135,7	2248,71
1550,78	19544	9359,4	320,7	3507,56	2422,63	2453,37	5964,33	2100,78
1434,22	19311,8	9263,4	316,81	3442,89	2278,8	2386,53	5965,52	2094,53
1681,98	21021,6	10499,8	330,64	3643,14	2483,86	2517,3	6147,97	2231,87
1555,41	19734,6	10187	316,47	3490,79	2402,02	2457,46	5925,22	2209,93
1471,04	20505,2	10514,5	334,39	3715,18	2541,63	2589,73	6229,02	2414,67
1547,43	21450,6	10812,9	337,23	3833,5	2668,07	2679,07	6601,37	2450,09
1419,67	19105,7	9267,2	327,41	3610,44	2607,5	2506,13	6688,49	2430,28
1413,94	20173,3	9859,1	354,57	3804,78	2904,47	2578,6	6914,19	2628,48
1593,3	22050,4	10806	360,75	4005,5	2885,76	2638,7	7077,48	2676,66
1576,86	22466,6	10850,8	369,13	4110,35	2895,82	2707,09	7272,32	2601,08
1535,19	21727,4	10576,5	365,62	3989,18	2882,18	2662,17	7041,31	2637,82
1434,65	22418	10878,9	359,94	4106,92	2846,05	2768,34	7514,46	2367,4
1309,46	21109,8	10476	349,44	4006,94	2787,38	2687,71	7293,69	2522,89
1279,06	20186,9	10359,9	339,65	3982,21	2766,73	2572,58	7376,24	2392,73
1204,15	18433,7	9630,7	329,22	3672,77	2611,32	2427,09	7158,77	2185,11
915,98	15563,2	8718,6	292,93	3256,76	2280,09	2267,88	5784,85	1948,44
798,42	14836,3	8546,6	280,18	2981,96	1947,85	2131,28	5502,02	1853,21
808,58	16017,7	8954,9	307,5	3242,84	1983,75	2139,18	6141,34	2027,15
682,21	15268,7	8449,5	299,68	3154,62	1846,91	2073,95	6088,84	1997,73
680,42	15089,7	8566,3	312,47	3159,81	1891,68	2083,42	5898,35	1942,06
796,02	15828	8509,2	318,47	3298,55	2076,49	2206,8	6458,91	2093,47

743,59	16351,4	8465,9	324,25	3452,45	2196,08	2275,86	6856,08	2243,63
728,93	15980,1	8008	323,51	3423,81	2159,06	2324,05	6946,83	2209,1
699,91	14592,3	7011	308,3	3212,8	2118,94	2208,44	6761,19	2087,76
525,45	12873,8	6089,8	290,09	3017,01	1897,04	2093,56	6264,38	1865,99
611,16	14274,4	7102,2	307,31	3196,65	1975,35	2227,63	6416,28	1871,67
598,68	13891	6738,1	326,47	3291,66	2014,8	2274,84	6772,26	1947,74
646,82	15100,5	7420,5	329,28	3413,07	2016,55	2345,69	6970,79	1994,91
739,12	15095,8	7708,5	323,18	3354,82	2089,74	2373,33	7216,15	2036,21
801,32	15539,7	7842,9	330,76	3429,27	2184,16	2369,21	7260,63	2052,3
809,14	15808,2	7934,6	336,55	3557,28	2301,99	2436,95	7405,5	2145,59
907,9	16273,4	8167,5	342,71	3641,07	2401,21	2475,81	7612,39	2210,02
986,76	17439,1	8362,3	354,35	3732,6	2446,04	2520,35	7776,05	2295,03
1007,99	15921,2	8230,3	340,53	3723	2466,6	2569,17	7741,7	2350,89
869,19	15338,7	7920	348,1	3731,42	2352,01	2592,19	7795,31	2324,36
974,09	16767,7	8419	351,39	3856,75	2414,25	2643,42	7913,71	2315,85
1014,53	17214,1	8320,6	363,38	3948,59	2416,69	2649,36	8348,84	2373,98
847,57	15239,3	7762,7	344,59	3738,91	2223,98	2526,11	7959,22	2220,67
884,6	16482,3	8433,4	369,81	3992,69	2337,74	2662,68	8275,97	2318,61
899,92	16682,2	8290,5	362,93	3933,78	2428,93	2673,42	8103,15	2379,75
1014,06	17434,9	9186,1	374,92	4143,44	2528,45	2802,27	8594,4	2630,54
1188,17	19351,5	9907,9	391,92	4299,89	2602,92	2904,35	9033,92	2745,18
1195,68	19021,5	9837,6	396,55	4295,21	2645,67	2870,89	9405,3	2816,36
1162,68	18967,7	9916,7	401,79	4295,95	2546,54	2923,82	9552,16	2835,17
1176,92	19418,3	9920,2	386,85	4165,72	2559,74	2891,25	9306,48	2718,7
1310,41	20442,4	10114,2	398,54	4408,08	2587,86	3096,91	9692,08	2922,03
1335,74	21691,9	10340,5	403,21	4391,5	2523,82	3129,94	9555,91	2843,44
1232,12	21783,4	10459	400,55	4487,39	2525,22	3089,8	9603,23	2842,35
1223,48	21629,7	10798,7	407,21	4519,57	2529,33	3159,1	9943,27	2954,6
1214,31	21283	10923,5	413,15	4422,84	2500,85	3127,21	9833,07	2920,18
1169,01	20570,8	10707,2	404,29	4246,14	2310,44	3098,74	9407,48	2927,48
1161,81	20450,5	10728,8	413,13	4381,04	2301,52	3192,72	9470,17	2938,18
1061,58	20892,1	10825,5	421,14	4416,24	2203,94	3221,4	9474,3	2939,29
915,83	19784	10477,8	411,32	4233,09	2214,7	3157,15	9326,87	2929,02
963,19	20014,8	10770,7	425,86	4390,18	2283,1	3287,91	9980,85	3041,41
826,18	19012	10279,5	406,17	4272,75	2160,08	3277,21	9805,55	2988,08

Furthermore are presented below the figures showing the movements of the price of each index through time.

Figure 5.1: Monthly prices chart of 5 years of ASE index.

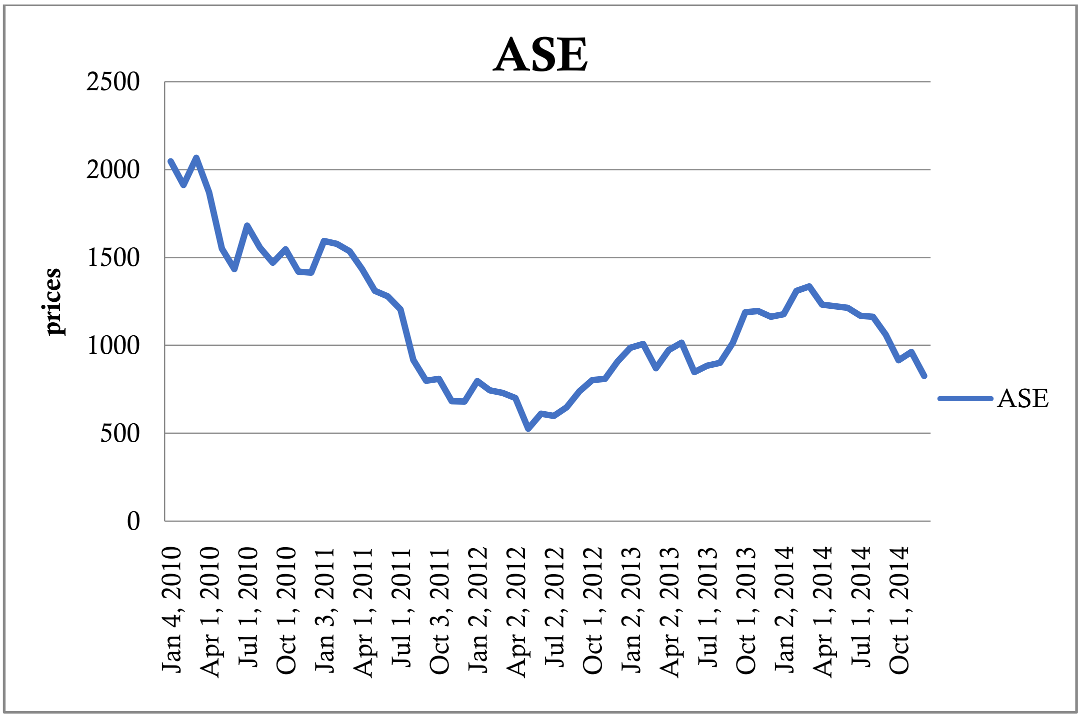


Figure 5.2: Monthly prices chart of 5 years of FTSE-MIB index.

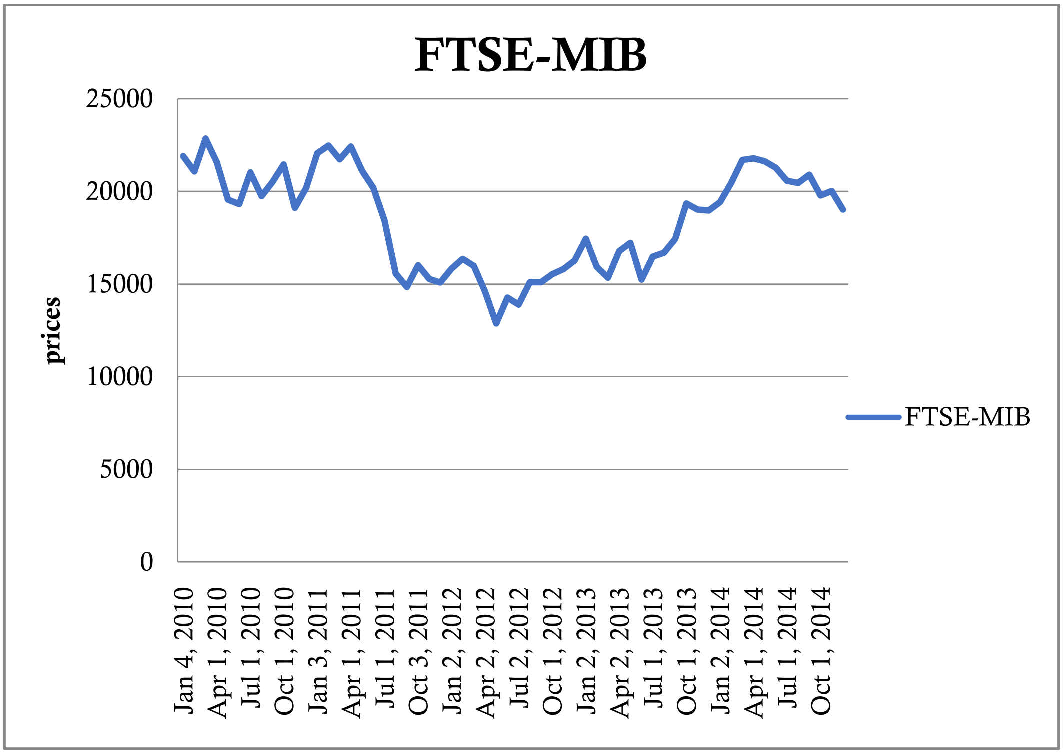


Figure 5.3: Monthly prices chart of 5 years of IBEX-35 index.

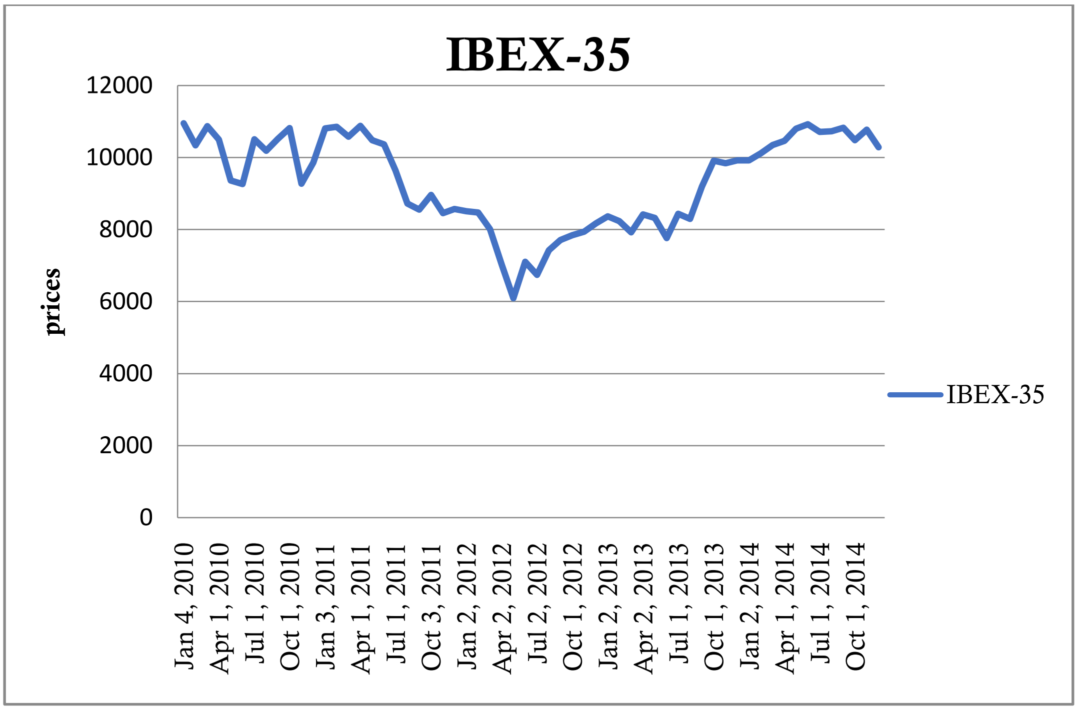


Figure 5.4: Monthly prices chart of 5 years of AEX index.

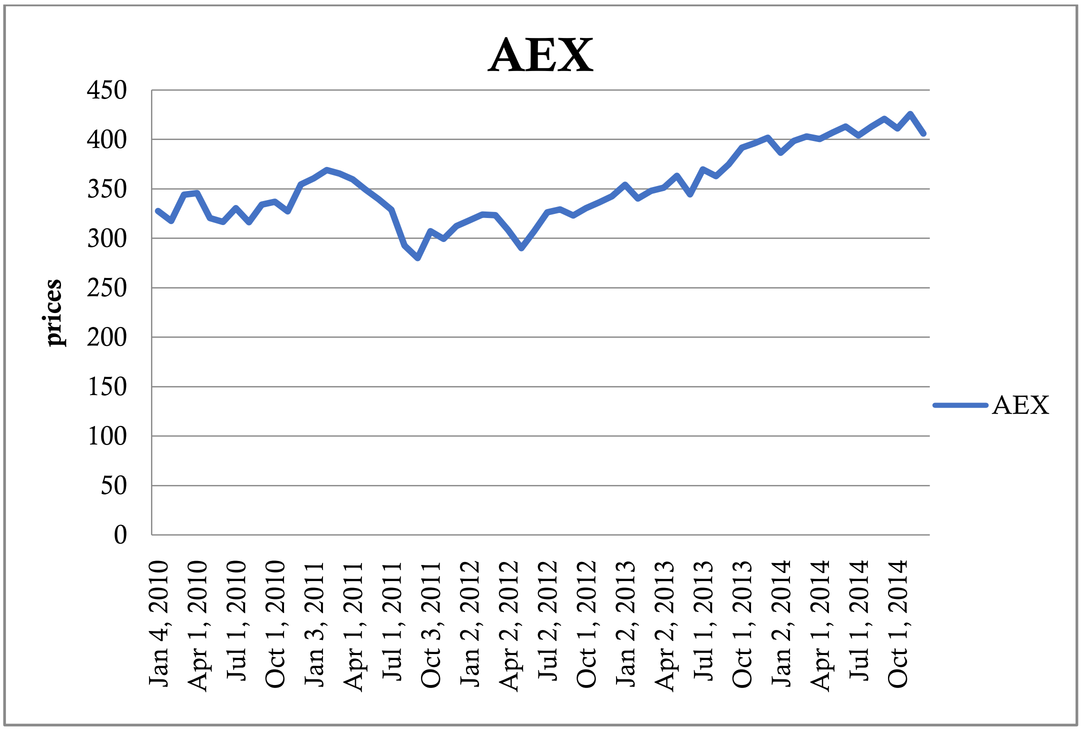


Figure 5.5: Monthly prices chart of 5 years of CAC-40 index.

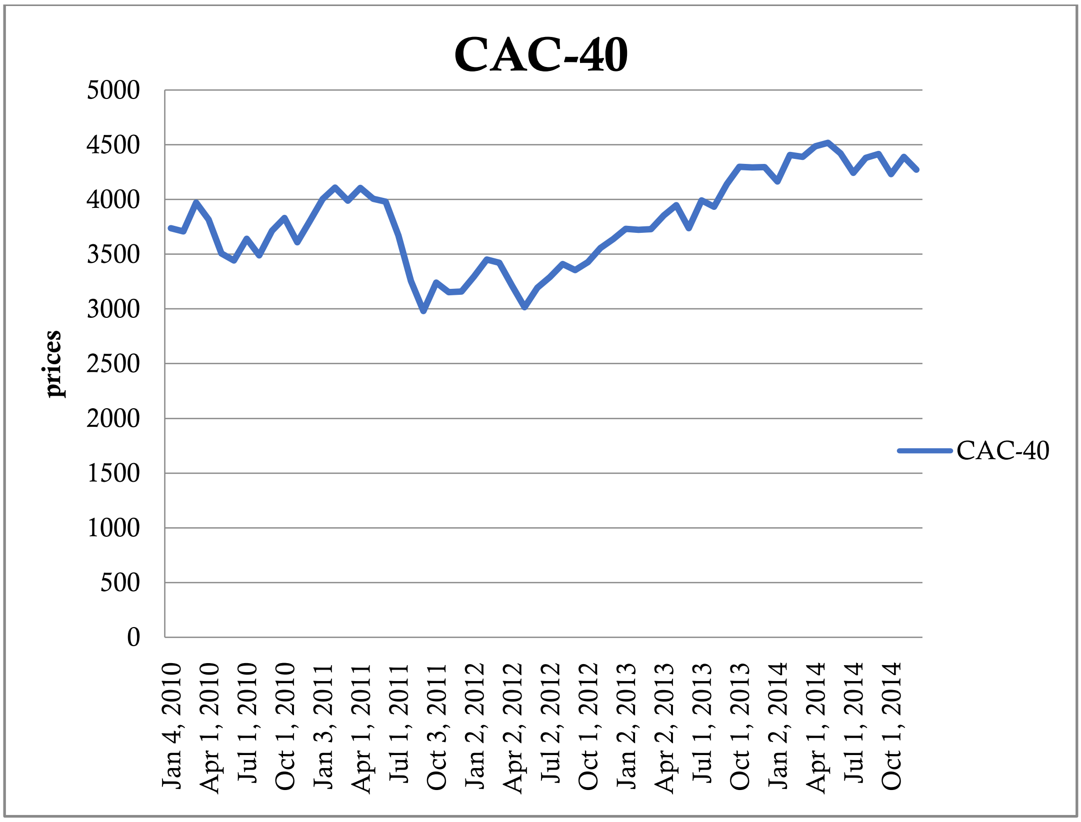


Figure 5.6: Monthly prices chart of 5 years of ATX index.

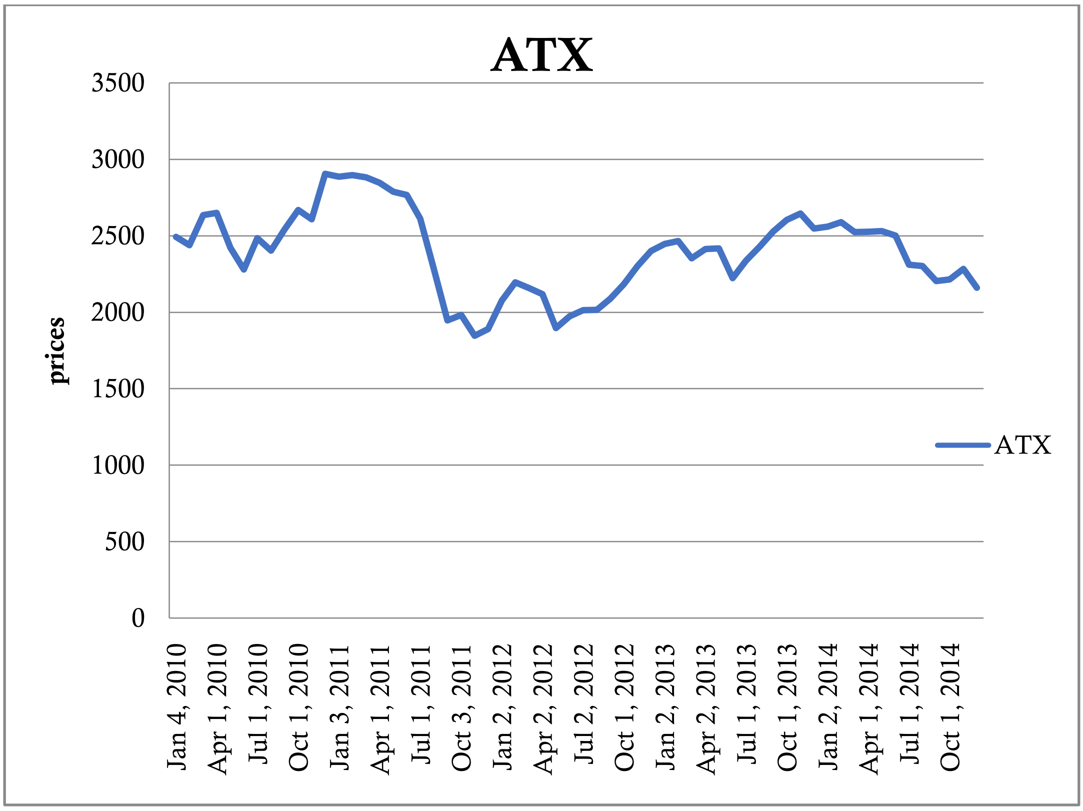


Figure 5.7: Monthly prices chart of 5 years of BEL-20 index.

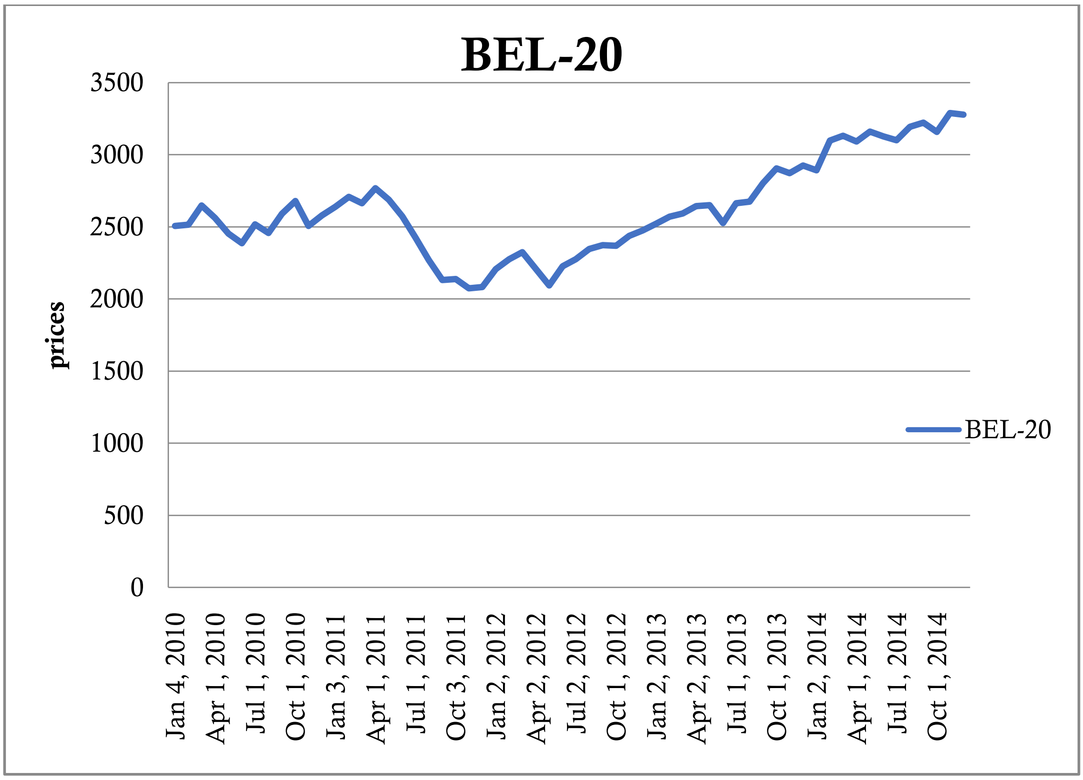


Figure 5.8: Monthly prices chart of 5 years of DAX index.

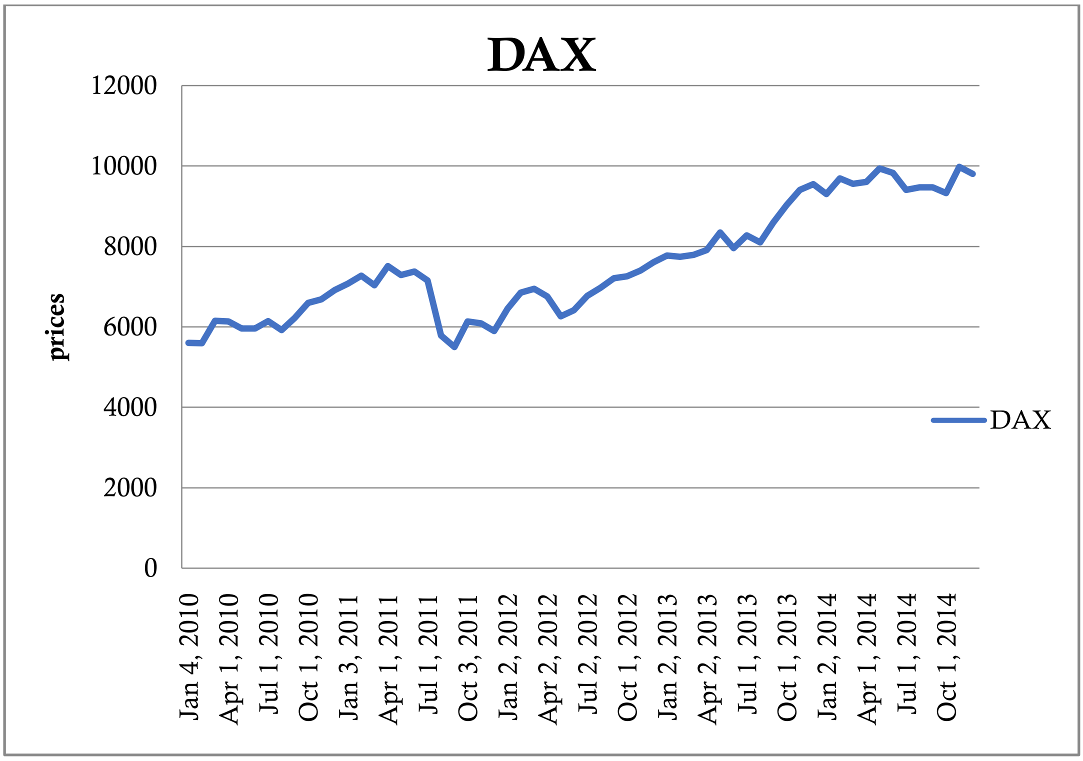
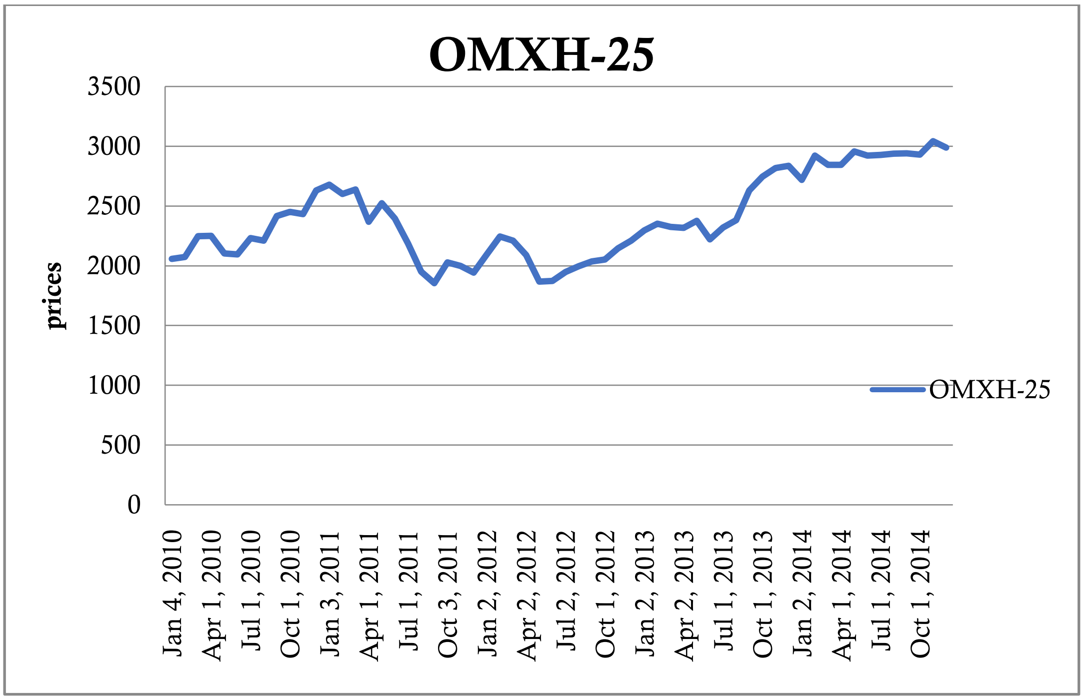


Figure 5.9: Monthly prices chart of 5 years of OMXH-25 index.



ASE is the Athens Stock Exchange General Index, which is a capitalization-weighted index of Greek stocks listed on the Athens Stock Exchange. The index was developed with a base value of 100 as of 31 December 1980. FTSE MIB is the benchmark stock market index for the Borsa Italiana, the Italian national stock exchange. FTSE MIB consists of the 40 most-traded stock classes on the exchange. IBEX -35 is the benchmark stock market index of Spain's principal stock exchange. It is a capitalization weighted index consisting by the 35 most liquid Spanish stocks traded in the Madrid Stock Exchange and its composition is reviewed twice per year. AEX index is a stock market index composed of Dutch companies that trade on the Amsterdam Stock Exchange. It is composed by 25 of the most actively traded securities on the exchange and it started from a base level of 100 index points on 3 January 1983. CAC-40 is a benchmark French stock market index. It is a weighted measure of the 40 most significant shares and it is reviewed quarterly by an independent Index Steering Committee. ATX is the Austrian traded index of the Vienna stock exchange and consists of 20 stocks. BEL-20 is the benchmark stock market index of the Brussels Stock Exchange consisting of a maximum of 20 stocks of 20 listed companies and its composition is reviewed annually. It is a European capitalization weighted index like the others. DAX is the German stock index consisting by 30 of the greatest German companies listed in the Frankfurt Stock Exchange and it started from 1000 as a base value. Finally OMXH-25 or else HEX25 is the stock market index of Helsinki Stock Exchange and like the others is a weighted index consisting of 25 traded stocks.

We also used a bond in the second portfolio that is going to be presented in this chapter. The bonds are a financial product, where an issuer borrows money at an interest and provide that at maturity will pay the lender the initial capital. The interest to the lender could be paid before the maturity in regular pre –agreed dates i.e. monthly, semi annually and annually. Bonds are separated in two categories, corporate and government bonds. In this thesis the role of the risk-free asset will get a government bond of 10 year duration with a constant yield 0,165% in a monthly basis for the needs of this application. As a risk free asset is used the German 10-year benchmark bond, so we assume that its yield is certain and equal to its interest rate $R_f = 0{,}165\%$ and the risk is zero $\sigma_f = 0$.

5.3. Portfolio Analysis of 8 stock indexes.

We will present the portfolio consisted of the following stock indexes: FTSE MIB, IBEX-35, AEX, CAC-40, ATX, BEL-20, DAX and OMXH-25. First are calculated the statistical measures that are needed for the mean-variance model with the use of the equations presented in the previous chapter that introduced the mean-variance analysis in a theoretical way. First we collected the monthly close prices shown above for the year 2010 until 2014 and we calculated the monthly return of each asset assuming that the investor purchased in the first month and proceeds to sell the second. We used the following formula:

$$R_t = LN(\frac{P_t}{P_{t-1}}) \quad (5.1)$$

Where R_t is the return of the stock index at the time period t-1 until t, P_t is the final price of the stock index at time t and P_{t-1} is the initial price at time t-1.

Then we calculated the expected return $E(R_t)$ of each stock index from the mean value of the historical monthly returns R_{it} by the equation:

$$E(R_t) = \frac{1}{n}\sum_{i=1}^{n} R_{it} \quad (5.2)$$

The results of this procedure are shown in the following table.

Table 5.2: Expected return of the stock indexes.

STOCK INDEX	EXPECTED RETURN
FTSE MIB	-0,239%
IBEX-35	-0,107%
AEX	0,363%
CAC-40	0,226%
ATX	-0,243%
BEL-20	0,455%

DAX	0,947%
OMXH-25	0,633%

The next step was to calculate the variance σ^2 and the standard deviation $\sqrt{\sigma^2}$ of the stock indexes returns by:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(R_{it} - E(R_t))^2 \qquad (5.3)$$

As a risk measure we use the standard deviation and the results are presented in the following table.

Table 5.3: Standard deviation of the stock indexes.

STOCK INDEX	**STANDARD DEVIATION**
FTSE MIB	6,370%
IBEX-35	6,150%
AEX	4,081%
CAC-40	4,508%
ATX	5,308%
BEL-20	3,524%
DAX	4,832%
OMXH-25	5,009%

After calculating the expected return and the standard deviation of the stock indexes we constructed the variance-covariance matrix with size 8×8 that contains diagonally the variances of the stock indexes and the elements outside the diagonal corresponds to the covariances of the stock indexes returns. The covariance of the returns of two assets shows the level of similarities between the variances of two assets. There are calculated eight variances and twenty eight covariances. The matrix is presented below and it is symmetrical.

Table 5.4: Variance-Covariance matrix.

STOCK INDEX	FTSE MIB	IBEX-35	AEX	CAC-40	ATX	BEL-20	DAX	OMXH-25
FTSE MIB	**0,399%**	0,340%	0,204%	0,246%	0,231%	0,180%	0,214%	0,195%
IBEX-35	0,340%	**0,372%**	0,173%	0,214%	0,199%	0,164%	0,159%	0,171%
AEX	0,204%	0,173%	**0,164%**	0,159%	0,156%	0,106%	0,150%	0,146%
CAC-40	0,246%	0,214%	0,159%	**0,200%**	0,186%	0,139%	0,177%	0,165%
ATX	0,231%	0,199%	0,156%	0,186%	**0,277%**	0,136%	0,182%	0,192%
BEL-20	0,180%	0,164%	0,106%	0,139%	0,136%	**0,122%**	0,119%	0,116%
DAX	0,214%	0,159%	0,150%	0,177%	0,182%	0,119%	**0,229%**	0,164%

OMXH-25	0,195%	0,171%	0,146%	0,165%	0,192%	0,116%	0,164%	**0,247%**

Another measurement more standardized that determines the degree to which two variable's movements are associated and has been theoretically explained in the previous chapter. Now we are going to present the correlations matrix since we have calculated the variances and the covariances we use the equation of the correlation coefficient.

$$\rho_{12} = \sigma_{12}/\sigma_1\sigma_2 \tag{5.4}$$

Table 5.5: Matrix of correlation coefficients of the stock indexes.

STOCK INDEX	FTSE MIB	IBEX-35	AEX	CAC-40	ATX	BEL-20	DAX	OMXH-25
FTSE MIB	**1,000**	0,884	0,797	0,871	0,696	0,816	0,709	0,622
IBEX-35	0,884	**1,000**	0,702	0,786	0,620	0,768	0,546	0,566
AEX	0,797	0,702	**1,000**	0,879	0,734	0,747	0,776	0,728
CAC-40	0,871	0,786	0,879	**1,000**	0,791	0,891	0,825	0,745
ATX	0,696	0,620	0,734	0,791	**1,000**	0,740	0,721	0,735
BEL-20	0,816	0,768	0,747	0,891	0,740	**1,000**	0,708	0,671
DAX	0,709	0,546	0,776	0,825	0,721	0,708	**1,000**	0,688
OMXH-25	0,622	0,566	0,728	0,745	0,735	0,671	0,688	**1,000**

The expected return of the portfolio consisting of these eight assets that each one has a weight w_i is given by:

$$E(R_p) = \sum_{i=1}^{8} w_i E(R_i) \tag{5.5}$$

The standard deviation σ_p of the portfolio is given by:

$$\sigma_p = \sqrt{\sum_{i=1}^{8} w_i^2\sigma_i^2 + \sum_{i=1}^{8}\sum_{j=1}^{8} w_i w_j \sigma_{ij}} \quad i \neq j \tag{5.6}$$

We have calculated or the variables that are needed for the calculation of the expected return and the standard deviation of the portfolio except from the weights of the assets that are going to be calculated so as for the portfolio to be optimal. They are going to be calculated by the procedure for the calculation of the efficient frontier that drives to the calculation of the optimal weights, the expected returns and standard deviations of the optimal portfolios.

In the next two figures are presented graphically in the form of a bar graph the expected returns and the standard deviations of the assets of the portfolio.

Figure 5.10: Expected returns of the stock indexes.

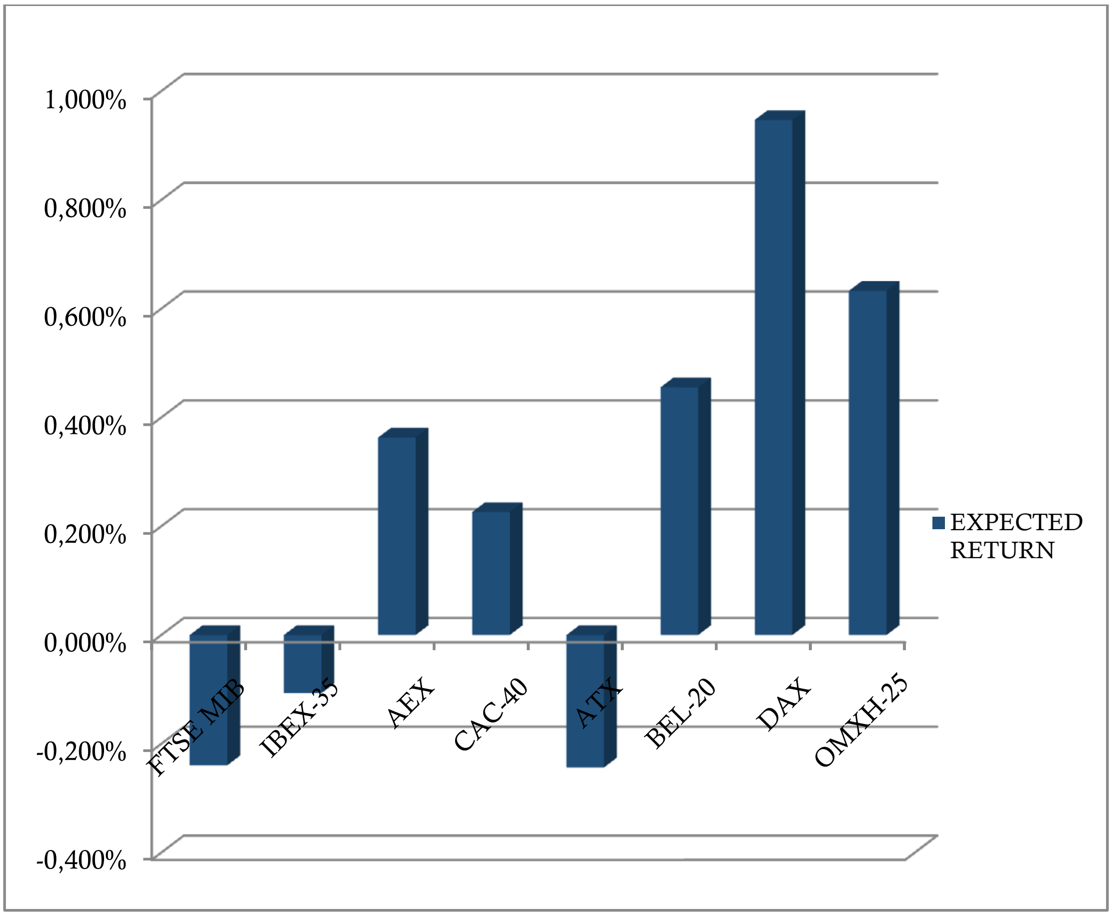
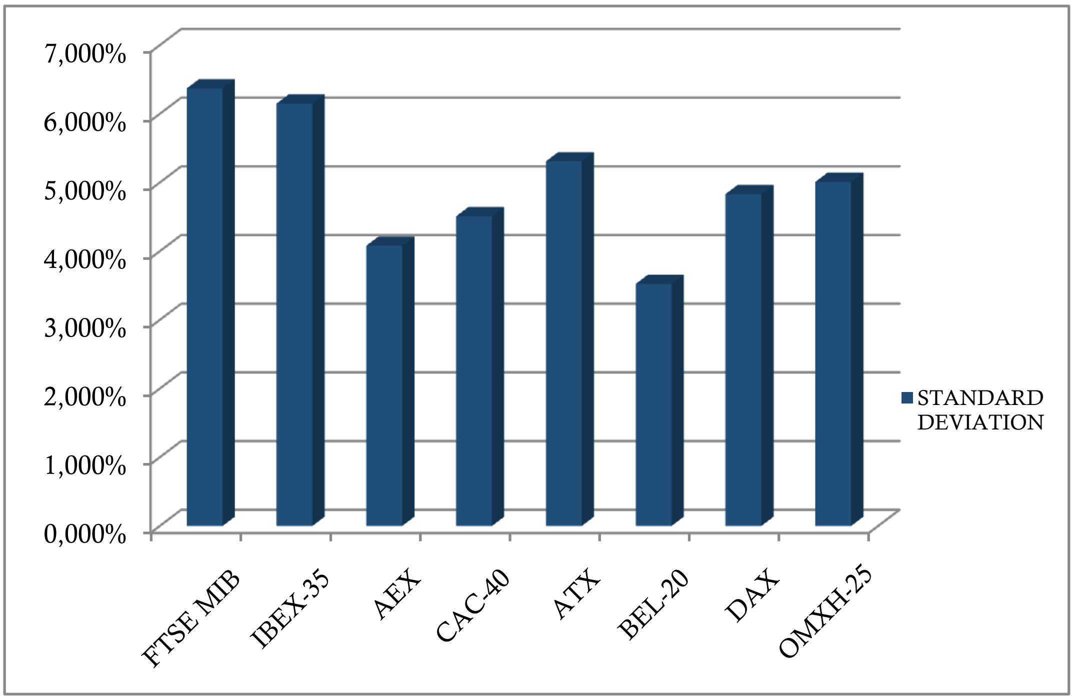


Figure 5.11: Standard deviation of the stock indexes.



As we can see in the Figure 5.10 the stock index with the greatest expected return is DAX and then follows OMHX-25 and BEL-20. Also countries such as Italy, Spain and Austria have indexes, FTSE MIB, IBEX-35 and ATX with negative expected return. As far as standard deviation we can see in Figure 5.11 that the indexes with the lowest risk are BEL-20 and AEX and with the highest risk are FTSE MIB and IBEX-35.

Now we are going to show how we calculated the efficient frontier and present its solution algorithm. The aim of this algorithm is to determine the participation rates of its asset, for each level of expected return-risk. We will then present the results of solving the problem of finding the efficient frontier and with the efficient frontier we will also define and present the minimum risk and maximum return portfolios.

The efficient frontier is defined from a set of combinations that minimize risk for every level of expected return. Therefore if we want to calculate any point located on the efficient frontier we minimize the risk of the portfolio under the constraints of a fixed level of expected return on the portfolio, of investing the whole capital on the portfolio and under the assumptions of the absence of short selling and of a risk-free asset. All this are expressed as follows:

Minimize: $\sigma_p^2 = \sum_{i=1}^{8} w_i^2 \sigma_i^2 + \sum_{i=1}^{8} \sum_{j=1}^{8} w_i w_j \sigma_{ij}$ with $i \neq j$

Subject to the constraints: $\sum_{i=1}^{8} w_i E(R_i) = E(R_P) = \overline{R_P}$ (1)

$$\sum_{i=1}^{8} w_i = 1 \qquad (2)$$

$$w_i \geq 0, i = 1, \ldots\ldots, 8 \qquad (3)$$

Finding the efficient frontier is an optimization problem where in the case of the absence of short selling and a risk free asset is a minimization problem solved by the use of quadratic programming with the help of the nonlinear solution method of the Excel's Solver Add-In.

First we determine the minimum variance portfolio, which means the one with the lowest risk (minimum standard deviation). To find it we use Solver and minimize the variance without setting a certain level of expected return under the constraints 2 and 3. The following figure shows the percentages of the stock indexes at the minimum risk portfolio and the table shows the statistical data.

Figure 5.12: Percentages of the stock indexes in the minimum variance portfolio.

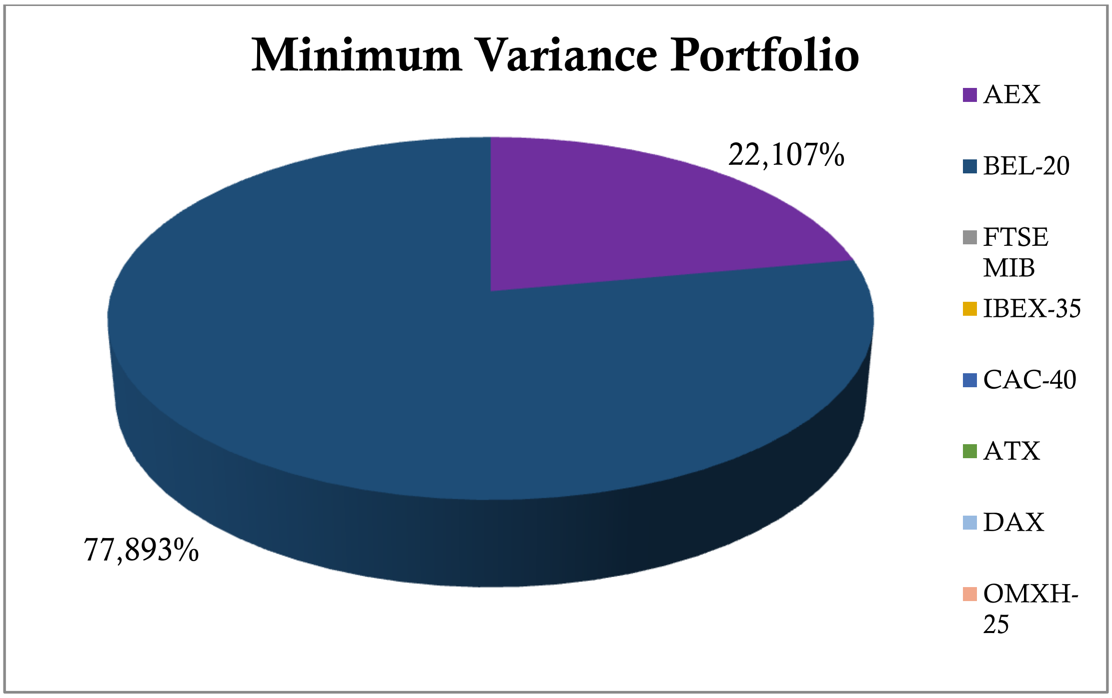


Table 5.6: Statistical data of the minimum variance portfolio.

MINIMUM RISK PORTFOLIO	EXPECTED RETURN	VARIANCE	STANDARD DEVIATION
	0,435%	0,118%	3,441%

As we noticed earlier the stock index BEL-20 has the smallest variance 0,124% and standard deviation 3,524% but the diversified portfolio has variance 0,118% and standard deviation 3,441%. The risk of the portfolio is smaller than the risk of the separate stock index as we expected due to the benefits of diversification. The minimum variance portfolio is the lower boundary of the efficient frontier given the assumptions and the constraints that we discussed earlier.

We are going to calculate the upper boundary of the efficient frontier, which is the maximum return portfolio. To define it we use Solver to maximize the expected return function, without setting any constraint for the function of the variance. The constraints that need to be satisfied so as to maximize the objective function are the constraints 2 and 3 that we defined earlier. The Solver Add-In finds the solution and the results of the percentages of the weights of the stock indexes that are concluded in the maximum return portfolio are presented in Figure 5.13 and the expected return and standard deviation of the maximum return portfolio are presented in Table 5.7. As we can see in the figure below the maximum return portfolio consists entirely of the more efficient stock index, DAX.

Figure 5.13: Percentages of the stock indexes of the maximum return portfolio.

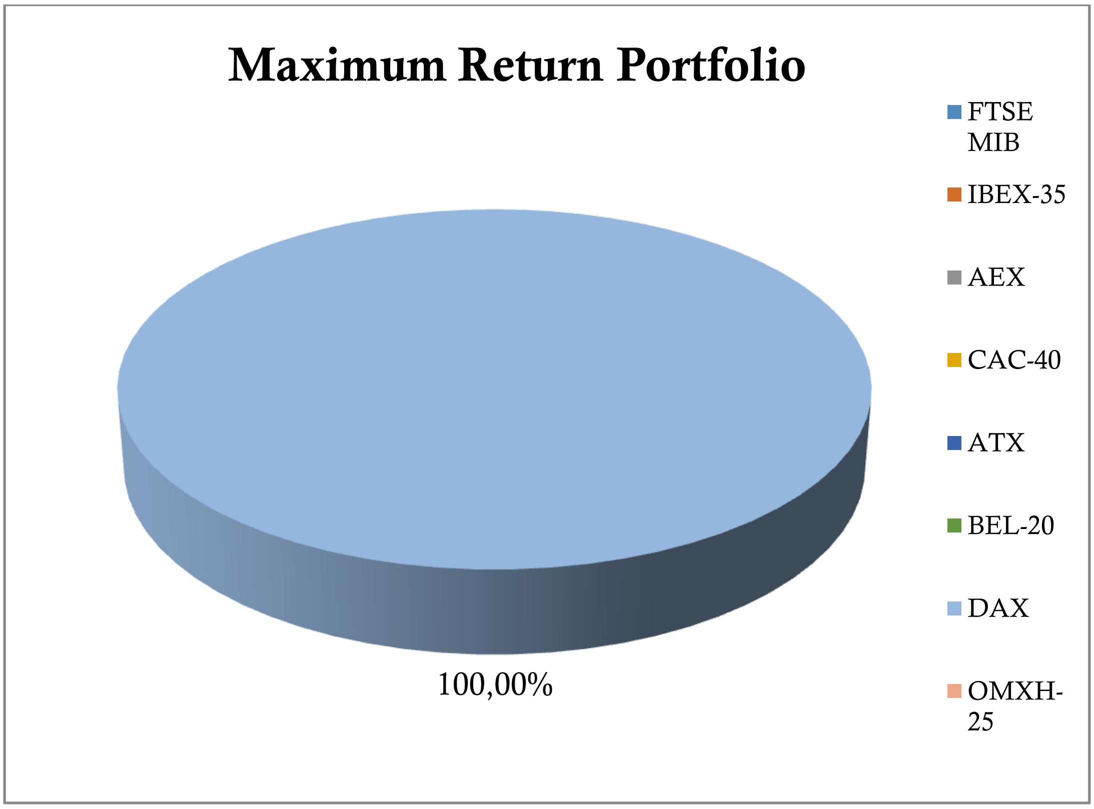
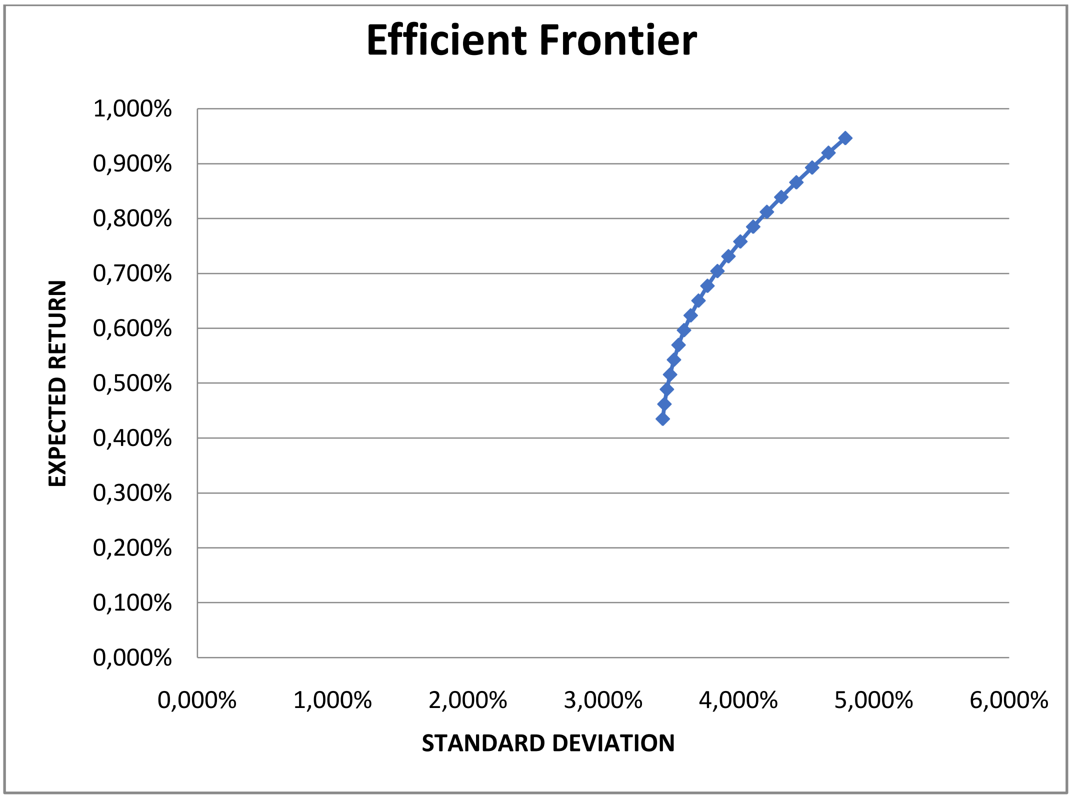


Table 5.7: Statistical data of the maximum return portfolio.

MAXIMUM RETURN PORTFOLIO	EXPECTED RETURN	VARIANCE	STANDARD DEVIATION
	0,947%	0,229%	4,790%

After calculating the minimum variance and the maximum return portfolios, namely the two ends of the efficient frontier, now depending on the number of portfolios that we want to design we will define a step according to which the expected return will increase each time. The step is defined as the difference between the expected returns of the two portfolios to the number of the portfolios we want to design. In this case we designed 19 portfolios and the step is: $E(R_{pMR}) - E(R_{pMV})/19 = 0{,}02695\%$. Then we calculate the efficient frontier starting from the lower boundary and repetitively increasing the step until we reach the top boundary. At each level of constant expected return we minimize the objective function of variance and solve through Solver the quadratic programming problem under the constraints 1, 2 and 3. We find in every solution an efficient portfolio, in specific we find each standard deviation and the expected return that is known and constant and the percentages of the optimal weights of each asset in every efficient portfolio that we designed. The efficient frontier that shows the relationship between risk and expected return is presented in the following figure.

Figure 5.14: Efficient frontier of the eight stock indexes.



The efficient frontier is a subset of feasible portfolios that are efficient and preferred from investors that are risk averse and prefer the largest expected return. In this case there is no short selling and no riskless lending and borrowing and that is why the efficient frontier is a concave function as shown in Figure 5.14 graphically.

Table 5.8: Optimal weights of the designed portfolios.

FTSE MIB	IBEX-35	AEX	CAC-40	ATX	BEL-20	DAX	OMXH-25
0,00%	0,00%	22,11%	0,00%	0,00%	77,89%	0,00%	0,00%
0,00%	0,00%	16,77%	0,00%	0,00%	78,75%	4,48%	0,00%
0,00%	0,00%	12,81%	0,00%	0,00%	77,97%	9,22%	0,00%
0,00%	0,00%	8,73%	0,00%	0,00%	77,08%	13,79%	0,40%
0,00%	0,00%	4,57%	0,00%	0,00%	76,12%	18,24%	1,07%
0,00%	0,00%	0,39%	0,00%	0,00%	75,16%	22,71%	1,74%
0,00%	0,00%	0,00%	0,00%	0,00%	70,38%	28,23%	1,39%

0,00%	0,00%	0,00%	0,00%	0,00%	65,19%	33,88%	0,94%
0,00%	0,00%	0,00%	0,00%	0,00%	59,99%	39,52%	0,49%
0,00%	0,00%	0,00%	0,00%	0,00%	54,80%	45,17%	0,03%
0,00%	0,00%	0,00%	0,00%	0,00%	49,34%	50,66%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	43,86%	56,14%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	38,37%	61,63%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	32,89%	67,11%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	27,39%	72,61%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	21,91%	78,09%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	16,43%	83,57%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	10,94%	89,06%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	5,46%	94,54%	0,00%
0,00%	0,00%	0,00%	0,00%	0,00%	0,00%	100,00%	0,00%

Every row of the Table 5.8 shows the optimal weights of each constructed efficient portfolio. Furthermore in Figure 5.15 we can see the efficient frontier of the eight stock indexes and each stock index separately. Every portfolio that is above the efficient frontier is not feasible and all the portfolios that create the separate stock indexes, are below the efficient front and they are feasible but they are not efficient, because in the same risk there is a point on the efficient frontier that has bigger expected return. Moreover we have calculated the expected return of ASE index -1,539% and the standard deviation 10,362% the same way as explained earlier for the other indexes. As shown in Figure 5.16 ASE index is below the efficient frontier and it is a feasible combination of risk-expected return, but is very far away from the efficient frontier and from the separate stock indexes. ASE index has very poor performance compared to all the other assets and has very high risk. That is something that we expected since Greece has been affected badly during this period of economic crisis.

We can also observe at the following graphs that other separate stock indexes that are of high risk and poor expected returns are FTSE MIB, IBEX-35 and ATX, those are the indexes of Spain, Italy and Austria, countries that have also been negatively affected from the economic crisis, since we study the period 01/01/2010-31/12/2014. Still Greece's index is the worst combination of expected return and risk and obviously very distant from the efficient frontier.

Figure 5.15: Efficient frontier of eight assets and each asset separately.

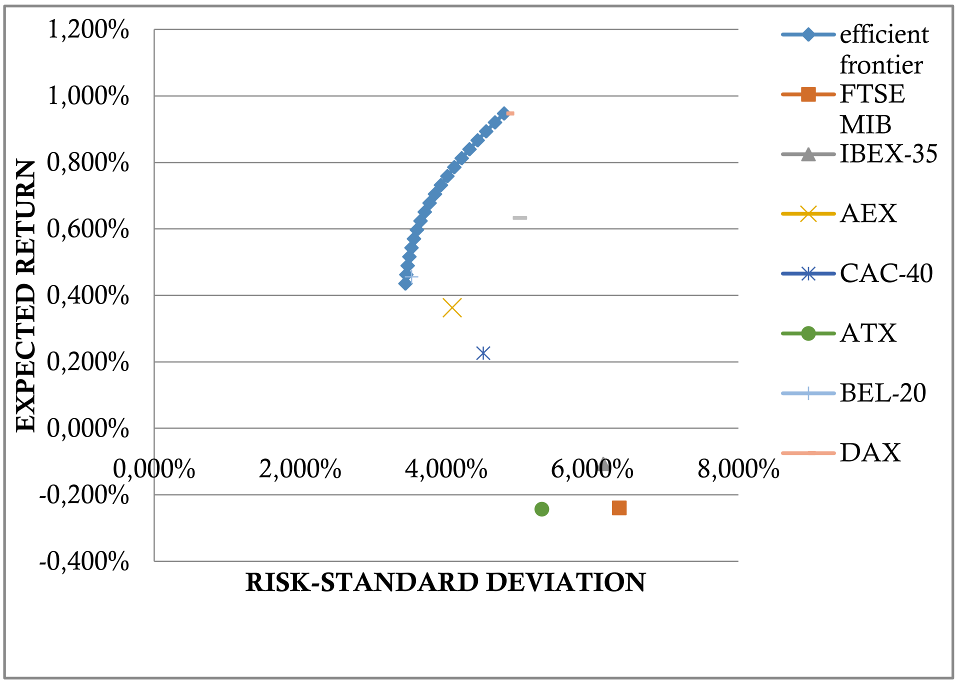
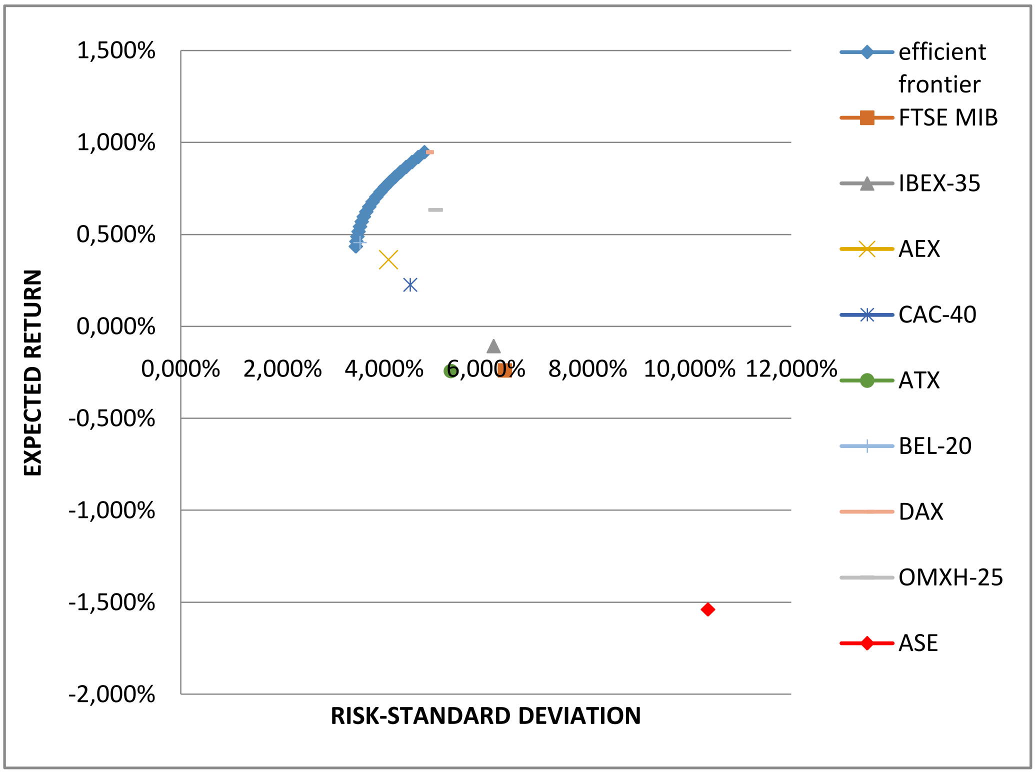


Figure 5.16: Efficient frontier, the eight stock indexes separately and ASE index.



5.4. Portfolio Analysis of 8 stock indexes after the input of the risk free asset.

We are going to show now how to calculate an efficient frontier of a portfolio that consists from the same eights stock indexes as the portfolio we designed before and from the German benchmark bond of 10 years that has a certain yield equal to $R_f = 0{,}165\%$ as we said earlier when we presented the data.

The combinations of the portfolio A of the eight stock indexes and of the bond with $R_f = 0{,}165\%$ creates a portfolio B, with weights w for the risky assets and 1-w for the risk-free asset. The expected return and standard deviation of this portfolio B is given by:

$$\overline{R_B} = w\overline{R_A} + (1\text{-w})R_F \tag{5.7}$$

$$\sigma_B = w\sigma_A \tag{5.8}$$

We can see that the risk σ_B is a percentage w of the risk σ_A of the risky portfolio and is smaller thanσ_A. By replacing w in the equation 5.7 that gives us the expected return of the new portfolio B we have:

$$\overline{R_B} = R_F + (\frac{\overline{R_A} - R_F}{\sigma_A})\sigma_B \tag{5.9}$$

In the case that short selling is not allowed and there is the possibility of investing in a risk-free asset the determination of the efficient frontier is based on the fact that the straight line joining the bond with the optimal portfolio is the one with maximum slope. The slope of the line is the ratio of excess return of the portfolio, which is the difference between the expected return of the portfolio B and the risk free rate, to the standard deviation of portfolio B. The mathematical representation of the quadratic programming problem to determine the efficient frontier is expressed as follows:

Maximize: $\frac{\overline{R_B} - R_F}{\sigma_B}$

Subject to: $\sum_{i=1}^{9} w_i = 1$

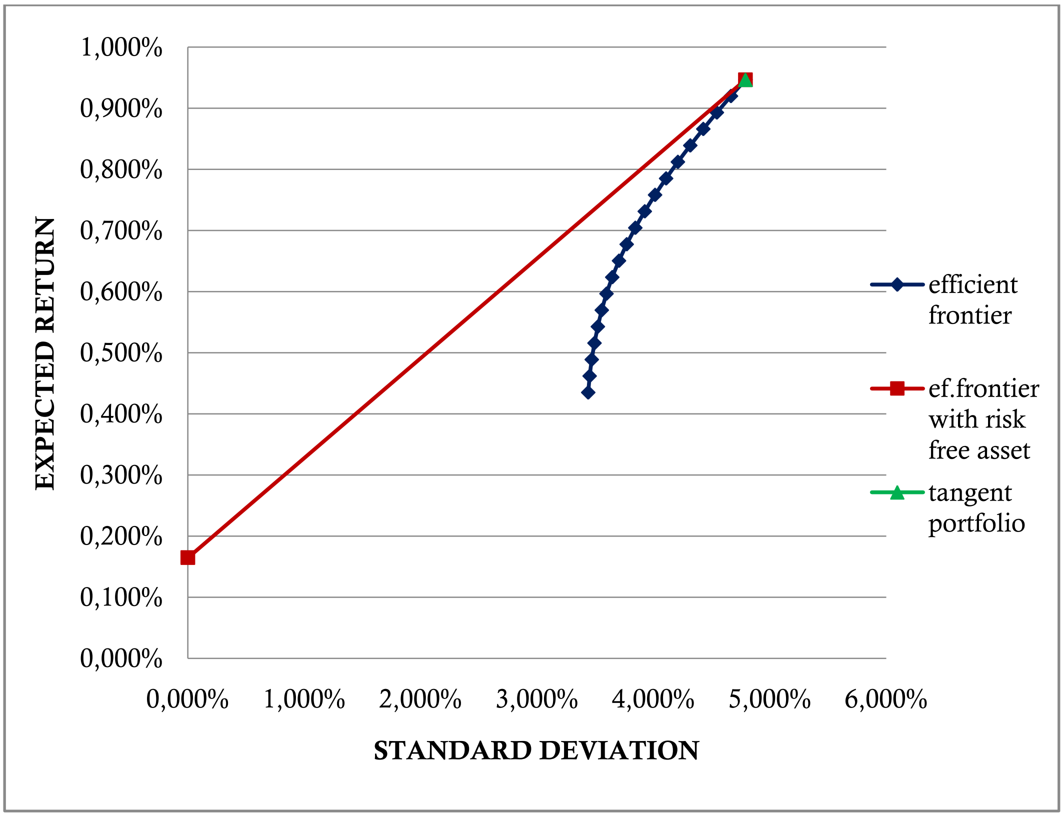
$w_i \geq 0, \qquad \forall i$

We used Solver to solve this non-linear programming problem and we found the tangent point of the line that is the portfolio's B efficient frontier, with the efficient frontier of the portfolio A, called tangent portfolio. This point maximizes the slope of the line and it is the maximum return portfolio, this portfolio consists entirely of the stock index with the greater expected return. We can see in the Table 5.9 the characteristics of the new efficient frontier and in Figure 5.17 the new efficient frontier and the efficient frontier of portfolio A.

Table 5.9. Characteristics of the new efficient frontier.

POINT OF THE FRONTIER	EXPECTED RETURN	STANDARD DEVIATION	PERCENTAGE OF THE BOND	PERCENTAGE OF THE STOCK INDEXES
STARTING POINT	0,165%	0,000%	100%	0%
FINAL POINT	0,947%	4,790%	0%	100%

Figure 5.17: The efficient frontier of the new portfolio.



We observe in Figure 5.17 that the new efficient frontier is a red straight line and not a curve like the efficient frontier of the portfolio A that consists only of risky assets. It starts from the minimum risk portfolio that consists only from the German benchmark bond and has a tangent point with portfolio B, which is a tangent portfolio that consists entirely from the DAX stock index. All the combinations on the efficient frontier of the portfolio B are better combinations of risk-return than the combinations on the efficient frontier of the risky asset portfolio A. The vertical projection of each portfolio of the efficient frontier of the portfolio A to the new efficient frontier of the portfolio B gives us greater expected return portfolio with the same risk. Similarly the horizontal projection of each portfolio of the curve efficient frontier in the new efficient frontier leads to a lower risk portfolio with the same risk as we can see in the Figure 5.17. The portfolio B with the bond and the eight stock indexes excels the portfolio A that consists only of the eight stock indexes, according to the mean-variance theory. Of course in this case also ASE index compared to the new efficient frontier is very far away and has a bad expected return-standard deviation relationship.

5.5. Conclusions

This dissertation tried to present ways of optimal portfolio management and presented the aspects that play an important role in the selection and optimization of a portfolio like the types of risk that an investor has to deal with when choosing a portfolio, the ways to measure risk, how to immunize a portfolio against risk and how to measure the performance of a portfolio. In this chapter based on the mean-variance method we optimized two portfolios, one only with risky assets and the other one with the risky assets and a risk free, found the efficient frontiers and presented the results of the research and compared the two frontiers. We used Excel and in specific the Solver Add-Inn to solve the quadratic programming problem.

We used European stock indexes and the German bond and showed that when the bond is included in the portfolio the efficient frontier in this case exceeds the efficient frontier of risky assets. Moreover we compared the Athens stock index with the portfolios that we constructed that of course were better than the ASE index. In conclusion although the mean-variance analysis that we used was introduced the first time in 1952 by Markowitz and extended by Tobin is until today an innovative methodology of choosing and composing efficient portfolios, able to carve a proper investment strategy.

Furthermore in the previous chapters of the dissertation we presented a review of literature concerning portfolio management through the years, in specific the modern portfolio theory and researches until recently and also another way of optimizing a portfolio more recent but a computationally difficult one, optimization with stochastic programming. Also we have presented extensively the research methodology, explaining all the tools that we used, the theoretical ones and the statistical, giving some important definitions and defining the constraints and the assumptions under which the research was made.

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