ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

ΤΜΗΜΑ ΟΙΚΟΝΟΜΙΚΗΣ ΕΠΙΣΤΗΜΗΣ και ΤΜΗΜΑ ΔΙΕΘΝΩΝ ΚΑΙ ΕΥΡΩΠΑΪΚΩΝ ΟΙΚΟΝΟΜΙΚΩΝ ΣΠΟΥΔΩΝ

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ΤΙΜΟΛΕΩΝ ΒΑΪΔΗΣ

**Διατριβή υποβληθείσα προς μερική εκπλήρωση
των απαραιτήτων προϋποθέσεων
για την απόκτηση του
Μεταπτυχιακού Διπλώματος Ειδίκευσης**

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Εγκρίνουμε τη διατριβή τού Τιμολέοντα Βαΐδη

ΝΙΚΟΛΑΟΣ ΤΟΠΑΛΟΓΛΟΥ

ΣΠΥΡΙΔΩΝ ΠΑΓΚΡΑΤΗΣ

ΓΕΩΡΓΙΟΣ ΟΙΚΟΝΟΜΙΔΗΣ

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Abstract

In this thesis the problem of portfolio management is addressed. The approach followed involves the minimization of the Conditional Value at Risk (CVaR) measure. A presentation of classes of risk measures pertinent to portfolio management is conducted and the motivation for utilizing the CVaR measure is documented. The implementation aspects of portfolio optimization schemes employing stochastic programming techniques are presented in detail. Option pricing techniques, fitted to the portfolio management algorithms are also presented, as an integral part of option based hedging tools. The performance of the presented portfolio optimization algorithms is investigated via numerical studies, involving an extended timeframe, ranging well before and after the major credit crisis of the years 2008-2009.

1 Introduction

Portfolio optimization has been a vastly visited issue in the literature of finance for the past several decades. Modern Portfolio Theory, mainly founded on the seminal work of Markowitz[1], addresses the problem of maximizing the expected return of a portfolio of assets for a given level of risk, or equivalently minimize risk for a given level of expected return, by appropriately deciding the portfolio assets quantities. In the previous decade a different approach in portfolio optimization has appeared dynamically in the finance literature, utilizing concepts from the field of risk management, combined with optimization algorithms within the class of stochastic programming[3]-[7]. The concept of coherent risk measures has been devised and exploited to characterize the risk aspects of multidimensional financial operational units more efficiently. The metrics of Value at Risk (VaR) and Conditional Value at Risk (CVaR) have become de facto industry standards for risk characterization.

Since the beginning of the last decade the CVaR risk metric has been dominantly employed in the derivation of algorithmic schemes for portfolio optimization. The derivation of efficient functional forms for its computation and its coherence properties (convexity, subadditivity) have rendered the CVaR a fundamental metric in the portfolio management research. The concepts and techniques presented in this document belong to the general class of CVaR based portfolio optimization algorithms.

The material presented in the document is outlined as follows.

In Chapter 2, a presentation of the theoretical framework of risk measures is provided. The concept of scenario generation, being the conceptual tool to translate the continuous time risk measures into discrete time and probability space, thus facilitating numerical implementation, is presented in detail. A variety of risk measures pertinent to portfolio of assets is presented analytically. Special emphasis is placed upon the VaR and CVaR functions. The special features of the CVaR function are documented both intuitively, via graphical representations and in a more rigorous fashion, via the concept of coherence for risk measures. In the last part of Chapter 2, a detailed analytical presentation for the computation of the CVaR measure, based on scenario generation is provided.

In Chapter 3, the central stochastic programming approach to the formulation of the portfolio optimization algorithm via CVaR minimization is presented analytically. The objective function and constraints for a portfolio of N assets are given and explained appropriately.

In Chapter 4, a sideway is introduced to present the issue of option pricing, in order to pave the way for the option hedged portfolios in the sequel. In the first part of the chapter, fundamental concepts of option pricing and certain option based hedging techniques are presented both analytically and via computer simulation examples, in order to document the relative merits of various option strategies for hedging, implemented in subsequent chapters. The scenario based option pricing is explained in detail. A separate section is devoted to the presentation of the risk neutral measure computation for scenario based option pricing. Such techniques have recently appeared in the literature[18], and fit as an integral part to optimization techniques for portfolios hedged via options. These techniques have been implemented via computer programs to produce the numerical results presented in chapter 6.

Having documented all the constituent parts of the option hedged portfolio optimization problem, Chapter 5 formulates and tabulates a specific algorithmic scheme for optimizing and backtesting the performance of a portfolio of stocks, hedged via put options. All the pertinent functions, variables, sets, parameters and equations are tabulated (CVaR minimization objective function, cash balance constraints, portfolio value expressions, backtesting initialization, etc).

IN Chapter 6, the results of computer studies are presented in detail. The stochastic programming algorithmic scheme tabulated in Chapter 5 is implemented via computer programs. The input data are produced from time series involving a multitude of stocks selected from the S&P500 index. Numerical results for several implementation scenarios are presented, regarding option hedging strategies, documented in previous chapters. Detailed performance comparisons among the hedging strategies are made. Special attention is drawn on results including the time period of the major credit crisis of 2008-2009. At the end of the chapter a separate section tabulates overall performance metrics of the various decision strategies.

Chapter 7 regards conclusions.

Finally, in Chapter 8 the listing of the computer programs implementing the algorithms presented throughout the thesis body is provided in the form of an Appendix .

2 Risk measures computation models

In the first part of this chapter the issues of risks related to portfolio management are generally addressed and commonly established risk measures are given. In the second part of this chapter, special emphasis is given on the percentile based risk measures, as they are judged more appropriate to effectively characterize the risk associated with flat tailed asset return distributions.

The concepts of VaR and CVaR are formulated. The discrete probability scenario methodology is also presented, as a means for the historical simulation approach for computing the VaR and CVar, which forms the basis for the portfolio optimization techniques presented in subsequent chapters.

In order to facilitate the definition of various risk metrics we first introduce the concept of scenario generation.

2.1 Scenario generation

We consider a portfolio consisting of N assets (e.g. stocks). Let $\mathbf{x} = (x_1,...,x_N)^T$ and $\mathbf{S}_0 = (S_{1,t},...,S_{N,t})$ denote the asset positions and prices at present time (t). Let $\mathbf{S}_{t'}$ denote the (uncertain) asset price at future time t. A scenario is employed in order to represent the uncertainty inherent in the random variables of the portfolio model. Due to the complex nature of the statistical entities and their interrelations, a formation of a scenario accurately reflecting their statistical properties is by itself a challenging task. A major objective of scenario generation algorithms is to render the scenarios appropriate for implementation in a stochastic programming framework for portfolio optimization, which is the central approach in this document, as well. Stochastic programming relies on discrete distributions of the involved random variables. An in-depth study of scenario generation so as to accurately reflect the statistical properties of the processes involved is not within the scope of this document. The reader is referred to the extensive work exposed in the literature on that subject (e.g., see [9] and references therein). A rather standard approach for a scenario generation[8], which is adopted in this work, is described below.

Fundamental to the scenario generation methodology is the adoption of discrete time analysis of the stochastic processes involved. Let t denote present time, counted in discrete time units (e.g. days, weeks, months, depending on the target time horizon of the portfolio manager). Assume a set of N_s discrete time observations of the portfolio asset values:

(1) $$\mathbf{S}_i = \{S_{i,t-N_{S+1}},\ldots,S_{i,t}\}, \quad 1 \le i \le N$$

where $S_{i,t-n}$ denotes the price of asset i at time t-n and N is the number of portfolio assets. The number N_S of past observations of asset values will be referred to as the "scenario size". Let $R_{i,t-n}$ denote the observed return ratios of the i-th asset at time t-n, i.e.,

(2) $$R_{i,t-n} = \frac{S_{i,t-n}}{S_{i,t-n-1}} \;, \; 1 \le n \le N_S - 1 \,, \quad 1 \le i \le N$$

Based on the sequence of observed return ratios, the scenario generated for the price of each of the N portfolio assets at time t+1 (the next time epoch of interest) is defined as the set Ω_S of $N \times N_S$ possible asset prices:

(3) $$\Omega_S = \left\{\tilde{S}_{i,n}\right\}_{n=1}^{N_S} = \left\{S_{i,t}R_{i,t-n}\right\}_{n=1}^{N_S} \;, \quad 1 \le i \le N$$

Equivalently, one can define the set Ω_R of $N \times N_S$ of possible asset returns

(4) $$\Omega_R = \left\{\tilde{R}_{i,n}\right\}_{n=1}^{N_S} = \left\{R_{i,t-n}\right\}_{n=1}^{N_S} \;, \quad 1 \le i \le N$$

As seen in (3), the scenario set at current time t (i.e., the set of possible future values of the portfolio assets at time t+1) is determined by the N_S recently observed historical returns of the portfolio assets as well as their currently observed prices. This is an intuitively satisfying definition, as it incorporates the statistical properties of the past in the possible determination of the immediate future. The scenario size (N_S) should be selected large enough to include possibly "unusually large" return variations, that might have happened some time in the history of each portfolio asset. These large return variations would result in the consideration of candidate future prices "distant enough" from its present value, as well. This is an important point, considering the flat-tailed distribution of assets and the issues associated with the necessity of introduction of the CVaR risk measure, as discussed in subsequent sections.

Naturally, a discrete probability measure should be assigned to scenarios at each time epoch t. If p_n denotes the "physical" probability of each scenario "leaf" happening, it should be that

(5) $$\sum_{n=1}^{N_S} p_n = \sum_{n=1}^{N_S} \mathrm{Prob}(S_{i,t+1} = \tilde{S}_{i,n}) = 1 , \quad 1 \le n \le N_S$$

The issue of scenario probability will be revisited in a subsequent section, when the risk neutral probability measure is discussed, in the context of option pricing.

2.2 Portfolio management risk measures

A rich classification of risks associated with the operations of financial institutions has been established in the finance community[20]: market risk, credit risk, foreign exchange risk, off-balance-sheet risk, country risk, technology risk, liquidity risk, insolvency risk, etc. It is not the objective of this document to address issues pertinent to all dimensions of risk analysis. The focus is on selected risk factors, associated with the management of a portfolio of assets (equities and options). The goal is to demonstrate the motivation for utilizing CVaR as a risk measure, upon which the portfolio optimization algorithms presented in subsequent chapters are built.

Having introduced the discrete time approach and the associated scenario set Ω_R of probable returns we proceed with the definition of various risk measures, for the portfolio consisting of N assets, with positions $\mathbf{x} = (x_1,...,x_N)^T$.

2.2.1 *Portfolio Variance*

The variance of the portfolio is given by

$$\sigma^2(\mathbf{x}) = \sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_{ij} x_i x_j \tag{6}$$

where σ_{ij} is the covariance of returns between assets i and j, defined as:

$$\sigma_{ij} = \frac{1}{N_S - 1} \sum_{n=1}^{N_S} p_n (\tilde{R}_{i,n} - \bar{\tilde{R}}_{i,n})(\tilde{R}_{j,n} - \bar{\tilde{R}}_{j,n}) \tag{7}$$

The notation $\bar{f}$ typically denotes average of the entity f. The multiplicative factor $1/(N_S - 1)$ simply renders σ_{ij} an unbiased estimator of the implied continuous time covariance of returns.

2.2.2 *Portfolio Semivariance*

We define the right and left semi-variances, respectively, as follows:

$$\hat{\sigma}^+(\mathbf{x}) = \sum_{n=1}^{N_S} p_n \sum_{i=1}^{N} (\max[\,(\tilde{R}_{i,n} - \bar{\tilde{R}}_{i,n}) x_i, 0])^2 \tag{8}$$

$$\hat{\sigma}^-(\mathbf{x}) = \sum_{n=1}^{N_S} p_n \sum_{i=1}^{N} (\max[\,(\bar{\tilde{R}}_{i,n} - \tilde{R}_{i,n}) x_i, 0])^2 \tag{9}$$

The right(left) semivariance measures upside(downside) deviations of the portfolio return from its mean value.

2.2.3 *Mean absolute deviation*

We define mean absolute deviation as follows:

(10) $$\hat{\omega}(\mathbf{x}) = \sum_{n=1}^{N_S} p_n \left| \sum_{i=1}^{N} (\tilde{R}_{i,n} - \overline{\tilde{R}}_{i,n}) x_i \right|$$

This risk measure penalizes deviations of the portfolio return from its mean, both on the upside and the downside. In direct analogy to the semivariance, to penalize portfolio return deviations from its mean only on the upside(downside) one can define the right(left) semi-absolute deviations as:

(11) $$\hat{\omega}^{+}(\mathbf{x}) = \sum_{n=1}^{N_S} p_n \max\left[\sum_{i=1}^{N} (\tilde{R}_{i,n} - \overline{\tilde{R}}_{i,n}) x_i , 0 \right]$$

(12) $$\hat{\omega}^{-}(\mathbf{x}) = \sum_{n=1}^{N_S} p_n \max\left[\sum_{i=1}^{N} (\overline{\tilde{R}}_{i,n} - \tilde{R}_{i,n}) x_i , 0 \right]$$

It can be proved that, if asset returns are normally distributed, the mean absolute deviation and variance risk measures are equivalent.

2.2.4 *Expected downside*

Assume that for each scenario we associate a random target return I_n. The expected downside is defined as

(13) $$G^{-} = \sum_{n=1}^{N_S} p_n \sum_{i=1}^{N} \max\left[0, I_n - \tilde{R}_{i,n}\right]$$

The expected downside risk measure is possesses some nice mathematical properties, as explained in section 2.4.

2.3 The VaR and CVar risk measures

As already mentioned, VaR is a measure of loss, answering the question: what is the maximum what loss with a specified confidence level, within a specified time horizon. We proceed with the mathematical formulation of the VaR definition.

We consider again the portfolio consisting of N assets (e.g. stocks), with position vector $\mathbf{x}=(x_1,...,x_N)^T$ and price vector $\mathbf{S}_0=(S_{1,t},...,S_{N,t})$ at present time (t). Let $\mathbf{S}_{t'}$ denote the (uncertain) asset price at future time t. The loss function $L(\mathbf{x},\mathbf{S}_t)$ is the difference between the current and future portfolio value at time t, i.e.,

(14) $$L(\mathbf{x},\mathbf{S}_{t'})=\mathbf{x}^T(\mathbf{S}_{t'}-\mathbf{S}_t)=\sum_{i=1}^{N}x_i(S_{t',i}-S_{t,i})$$

Let $p(\mathbf{S}_{t'})$ denote the joint probability density function of the multivariate stochastic entity $\mathbf{S}_{t'}$. For a given real number ζ, with $0\le\zeta\le1$, the probability $\Psi(\zeta;\mathbf{x})$ of the loss function being less than ζ is then

(15) $$\Psi(\zeta;\mathbf{x})=\mathrm{Prob}(L<\zeta)=\int_{-\infty}^{\zeta}L(\mathbf{x},\mathbf{S}_{t'})p(\mathbf{S}_{t'})d\mathbf{S}_{t'}$$

Let α be a fixed real number, which we call the confidence interval. Then, the VaR function is the a-percentile of the portfolio loss probability distribution at time t, i.e., it is this number $\zeta(\mathbf{x},\alpha)$ for which the cumulative probability of portfolio loss is less than the threshold a:

(16) $$\begin{gathered}VaR_a=\zeta(\mathbf{x},\alpha):\mathrm{Prob}(L<\zeta)=\alpha\Leftrightarrow\\ \alpha=\int_{L(\mathbf{x},\mathbf{S}_{t'})<VaR_\alpha}L(\mathbf{x},\mathbf{S}_{t'})p(\mathbf{S}_{t'})d\mathbf{S}_{t'}\end{gathered}$$

i.e.

(17) $$\mathrm{Pr}\,ob(L<VaR)=\alpha$$

Since a cumulative distribution is a monotone increasing function (practically genuinely monotone increasing), then VaR_a is actually the smallest number such that the probability of portfolio loss is less than the threshold *a*. A graphical representation of VaR is shown in Figure 1.

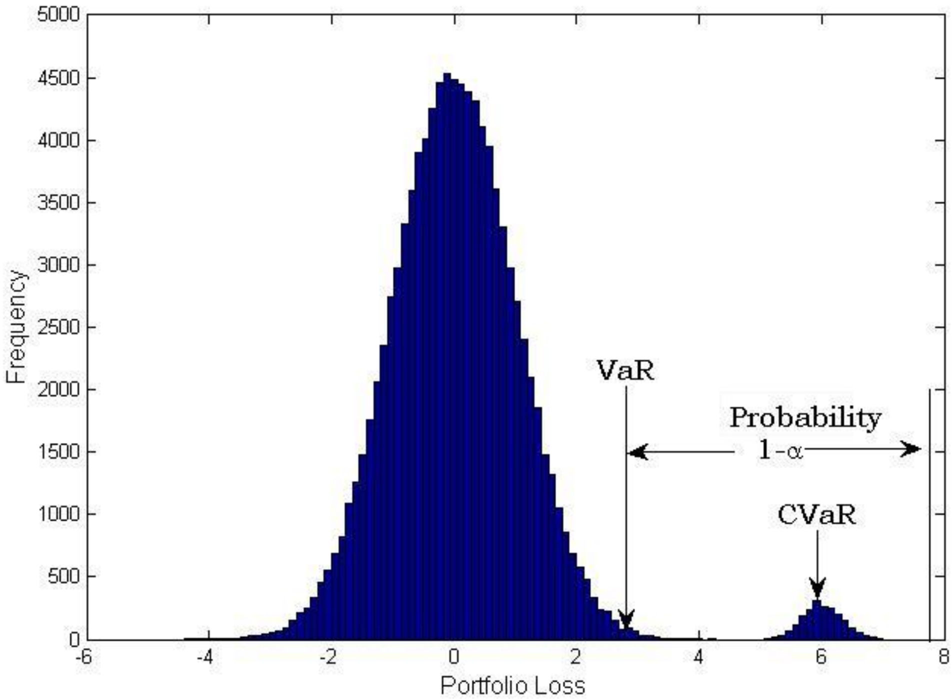

Figure 1: Graphical representation of VaR

As a risk measure, VaR offers an intuitively simple and consequently appealing concept for characterizing portfolio risk. This is the main reason why it is the dominant risk measure in the risk management community. However, as can be seen in Figure 1, one observes that the tail of the loss distribution can exhibit a more or less significant area, meaning that although we have concluded that loosing more than VaR is only 1-a probable, the amount of loss can still be significant. For the same VaR, depending on the shape of the Loss distribution function tail, there can be significantly different amounts of loss. Besides, it is possible that the VaR of a portfolio can be grater than the sum of VaR's of subportfolios, which is in intuitive contrast with the principle of diversification in finance.

The CVaR measure covers some of the caveats of Var. The Conditional VaR measure is defined as the conditional expected loss, given that the loss exceeds VaR, i.e.,

(18) $$CVaR(\alpha;\mathbf{x},\mathbf{S}_{t'}) = E(L/L > VaR) = \frac{E(L)}{\mathrm{P}(L > VaR)} = \frac{1}{1-\alpha}\int_{L>VaR} L(\mathbf{x},\mathbf{S}_{t'})\mathbf{p}(\mathbf{S}_{t'})d\mathbf{S}_{t'}$$

CVaR answers the question: how bad are losses expected to be if we are faced with losses heavier than VaR. This obviously entails the conclusion that Loss distributions with the same VaR can have very different CVaR. The distribution with the greater CVaR would inflict heavier losses in a portfolio, in case of a "bad" scenario.

2.4 Coherent Risk Measures

Having presented and defined portfolio management risk metrics, we present the concept of coherence in risk measurement metrics. It is by no means the objective of this section to exhaust the theoretical depth of this issue, but rather to substantiate the "appropriateness" of the CVaR as a risk measure for portfolio management, from a somewhat more strict theoretical point of view. For in depth coverage on the subject of coherence risk metrics the interested reader is referred to [21].

Consider two random variables X and Y (e.g. two portfolio asset losses). A coherent risk measure is a function φ assigned to each of these variables a measure (real number), such that:

(19) $$\begin{array}{ll} \phi(X+Y) \le \phi(X)+\phi(Y) & \text{subadditivity} \\ \phi(\lambda X) = \lambda\phi(X) & \text{homogeneit y} \\ X \le Y \Rightarrow \phi(X) \le \phi(Y) & \text{monotonicity} \\ \phi(X\text{-}nr_f) = \phi(X)\text{-}n & \text{risk free condition} \end{array}$$

In the expressions above the inequality $X \le Y$ means that for almost all possible distinct scenarios for the two random variables, we have $\tilde{X}_n \le \tilde{Y}_n$. The term $X\text{-}nr_f$ denotes losses are reduced by investing a positive amount n in the risk free asset r_f .

The subadditivity requirement for a risk measure is that, this allows measuring the risk of distinct constituents of a portfolio (or any financial entity) in an efficient manner. If a risk measure does not

satisfy the subadditivity requirement, it could be that, for example, the measured risk of individual portfolio assets (e.g. a 99,9%-VaR of stock losses) is zero, while the measured risk of the whole portfolio is not. This can potentially derail risk management.
The homogeneity and monotonicity properties ensure that the risk measure is convex, which guarantees the existence of global minimum. This is obviously a critical property for the formulation of risk-based portfolio optimization algorithms.
It can be shown that all risk measures presented do not satisfy the coherence properties, except CVaR and the Expected Downside.

The coherence properties of the CVaR, along with the development of a linear functional to compute it, as analyzed in the sequel, have led to the adoption of CVaR as the risk measure of choice for portfolio optimization algorithms.

2.5 Computing a Portfolio VaR and CVaR

Several approaches for the computation of the these risk measures have appeared in the literature[8]. The approach followed in this document belongs to the class of "historical simulation".
The concept of a "scenario" is central to the formulation of the theoretical justification as well as the implementation aspects of the VaR and CVaR computation.

2.5.1 *CVaR computation based on scenario generation*

Having discussed the scenario generation concept in the previous section, we return to the issue of Var and CVar computation. Assuming a portfolio of N assets, as in section 2.3, the Loss function under scenario n is expressed as:

(20) $$L(\mathbf{x},n)=\sum_{i=1}^{N}x_i(S_{i,t}-\tilde{S}_{i,n})=\mathbf{x}^T\left(\mathbf{S}_t-\tilde{\mathbf{S}}_n\right)\ ,\ \ 1\le n\le N_S$$

where $\mathbf{S}_t=(S_{1,t},...S_{N,t})^T$ is the vector of prices of portfolio assets at time t and $\tilde{\mathbf{S}}_n=(\tilde{S}_{1,n},...,\tilde{S}_{N,n})$ is the n-th scenario vector for portfolio assets at time t. The portfolio VaR at time t, with confidence interval a and time horizon 1 (discrete time) is the number ζ such that:

(21) $$\begin{aligned}VaR_a=\zeta(\mathbf{x},\alpha):\alpha&=\sum_{n:L(\mathbf{x},n)<\zeta}L(\mathbf{x},n)p(\tilde{\mathbf{S}}_n)=\sum_{n:L(\mathbf{x},n)<\zeta}\mathbf{x}^T(\mathbf{S}_t-\tilde{\mathbf{S}}_n)p(\tilde{\mathbf{S}}_n)=\\&=\sum_{n:L(\mathbf{x},n)<\zeta}\sum_{i=1}^{N}x_i(S_{i,t}-\tilde{S}_{i,n})p(\tilde{S}_{i,n})\end{aligned}$$

Employing the scenario description, the CVar measure of the portfolio is computed as

(22) $$\begin{aligned}CVaR=\frac{E(L)}{\mathrm{P}(L>VaR)}&=(1-\alpha)^{-1}\sum_{n:L(\mathbf{x},n)>VaR_\alpha}L(\mathbf{x},n)p(\tilde{\mathbf{S}}_n)\\&=(1-\alpha)^{-1}\sum_{n:L>VaR_\alpha}\mathbf{x}^T(\mathbf{S}_t-\tilde{\mathbf{S}}_n)p(\tilde{\mathbf{S}}_n)\\&=(1-\alpha)^{-1}\sum_{n:L>VaR_\alpha}\sum_{i=1}^{N}x_i(S_{i,t}-\tilde{S}_{i,n})p(\tilde{S}_{i,n})\end{aligned}$$

An important observation is due for the computation of CVar: as can be seen in expressions (21) and (22), the computation of the CVaR measure cannot be explicitly performed unless the highly non-linear VaR measure is known. This observation has motivated the development of considerably more efficient functional forms for CVaR computation[10], facilitating the introduction of a certain class of stochastic programming techniques for portfolio optimization, as discussed in detail in the sequel.

The approach followed in [10] was to construct a functionally similar to (18) but actually much simpler metric for the computation of CVaR, as follows:

(23) $$F_\alpha(\mathbf{x},\zeta)=\zeta+(1-\alpha)^{-1}\int_{L(\mathbf{x},\mathbf{S}_{t'})>\zeta}(L(\mathbf{x},\mathbf{S}_{t'})-\zeta)\mathbf{p}(\mathbf{S}_{t'})d\mathbf{S}_{t'}$$

where, instead of the VaR measure $\zeta(\mathbf{x},\alpha)$, the entity ζ is simply a real number and not a function of the problem parameters. It is shown in [10] that this functional has certain remarkable properties: i) it is convex with respect to the parameter ζ, ii) VaR is a minimum point of this function with respect to ζ, iii) minimizing $F_\alpha(\mathbf{x},\zeta)$ with respect to ζ yields CVaR, i.e.,

(24) $$CVaR_a = \min_{\zeta} F_a(\mathbf{x},\zeta)$$

This is a key conclusion, as it allows for direct computation of the CVaR risk measure, without prior knowledge of VaR. *Actually, it is shown that the value of ζ that minimizes* $F_\alpha(\mathbf{x},\zeta)$ *is the VaR itself*. Furthermore, minimization of the function $F_\alpha(\mathbf{x},\zeta)$ with respect to both $\mathbf{x}$ and ζ, optimizes CVaR and finds VaR simultaneously. It is exactly this key conclusion that has been exploited to perform portfolio optimization, based on CVaR, via stochastic programming techniques, as explained analytically in the next chapter.

3 Portfolio optimization based on CVaR minimization

In this chapter the risk measure properties and computation techniques presented in the previous chapter will be exploited to formulate portfolio optimization approaches based on CVaR minimization.

3.1 Portfolio optimization subject to Risk constraints

Financial institutions' typical approach in risk management is to estimate and control VaR at specified confidence intervals a, depending on their risk tolerance objectives. Since VaR is non-convex with respect to portfolio positions $(\mathbf{x})$, and it can exhibit multiple local minima, the problem with optimizing portfolios via VaR control can be really hard to solve. In contrast, posing CVaR constraints and replacing the CVaR function with the functional form (23) can transform the portfolio optimization problem to forms conveniently solvable using linear programming techniques. Since CVaR is a more conservative risk measure than VaR, $(CVaR \geq VaR)$, posing constraints on CVaR also restricts VaR. For example, computing the portfolio position $(\mathbf{x})$ that minimizes a metric $d(\mathbf{x})$, such as the mean loss $d(\mathbf{x}) = E(\mathbf{x}, \mathbf{S}_{t'})$, subject to the constraint that the $CVaR_\alpha$ at a certain confidence level a is less than a certain threshold C_α, is formulated as:

(25) $$\min_{\mathbf{x}} d(\mathbf{x})$$

such that

(26) $$CVaR_\alpha \leq C_\alpha$$

It is easily shown that this setup can be equivalently transformed to:

(27) $$\min_{\mathbf{x}} d(\mathbf{x})$$

such that

(28) $$F_\alpha(\mathbf{x}, \zeta) \leq C_\alpha$$

Indeed, if constraint (28) is satisfied for some ζ, then it is satisfied for that ζ^* that minimizes the functional $F_\alpha(\mathbf{x},\zeta)$. But, due to (24), $CVaR_a = \min\limits_{\zeta} F_a(\mathbf{x},\zeta) = F_a(\mathbf{x},\zeta^*)$. Therefore, constraint (28) is equivalent to constraint (26).

In the next section, we present the stochastic programming approach to the portfolio optimization based on CVaR minimization, utilizing the discrete probability distribution scenarios described in the previous chapter.

3.2 Stochastic programming portfolio optimization via CVaR minimization

A considerable body of literature has been produced in the last decade on the subject of portfolio optimization via stochastic programming techniques[10]-[14]. In this section this optimization methodology is presented in detail.

Following the notation of chapter 1, we consider a portfolio consisting of N assets (e.g. stocks). Let $\mathbf{x} = (x_1,\ldots,x_N)^T$ and $\mathbf{S}_0 = (S_{0,1},\ldots,S_{0,N})$ denote the asset positions and prices at present time (t). The portfolio loss expression is repeated here for convenience:

$$L(\mathbf{x},n) = \sum_{i=1}^{N} x_i (S_{i,t} - \tilde{S}_{i,n}) = \mathbf{x}^T \left(\mathbf{S}_t - \tilde{\mathbf{S}}_n\right),\ 1 \le n \le N_S$$

where $\mathbf{S}_t = (S_{1,t},\ldots S_{N,t})^T$ is the vector of prices of portfolio assets at time t and $\tilde{\mathbf{S}}_n = (\tilde{S}_{1,n},\ldots,\tilde{S}_{N,n})$ is the n-th scenario vector for portfolio assets at time t. Considering a confidence level a, the CVaR functional in (23) is written as

$$
\begin{aligned}
F_\alpha(\mathbf{x},\zeta) &= \zeta + (1-\alpha)^{-1} \sum_{n:L(\mathbf{x},\tilde{\mathbf{S}}_n)>\zeta} [L(\mathbf{x},n)-\zeta]\mathbf{p}(\tilde{\mathbf{S}}_n) \\
&= \zeta + (1-\alpha)^{-1} \sum_n [L(\mathbf{x},n)-\zeta]^+ \mathbf{p}_n \\
&= \zeta + (1-\alpha)^{-1} \sum_n [\mathbf{x}^T(\mathbf{S}_t - \tilde{\mathbf{S}}_n) - \zeta]^+ \mathbf{p}_n
\end{aligned} \tag{29}
$$

where $\mathbf{p}_n$ is the discrete probability of the n-th senario and $f^+ = \max(f,0)$. At this stage, an auxiliary variable y_n is introduced to represent the positive deviations of the loss function from the VaR measure, as follows:

$$y_n \geq L(\mathbf{x},n) - \zeta \tag{30}$$

along with the constraint

$$y_n \geq 0 \tag{31}$$

Then, the minimization of $F_\alpha(\mathbf{x},\zeta)$ is equivalent to minimizing the linear expression

$$\min_{\mathbf{x},\zeta} \zeta + (1-\alpha)^{-1} \sum_n p_n y_n \tag{32}$$

subject to the constraints (30), (31) and the definition of the loss function (20). This is a constraint optimization problem solvable via linear programming.

Additional constraints can be imposed, such as maximum allowable CVaR

$$\zeta + (1-\alpha)^{-1} \sum_n p_n y_n \leq C_\alpha \tag{33}$$

where C_α is a maximum threshold allowed for the CVaR measure. Alternatively, a minimum target return on assets can be imposed (equivalently maximum on losses), as follows

$$\sum_{n=1}^{N_S} L(\mathbf{x},n) p_n \leq -\mu \tag{34}$$

where μ should be a negative number (the negative of minimum targeted return).

In a subsequent section the stochastic programming methodology presented above is applied to construct a detailed model for optimization of a portfolio consisting of stocks and options. However, to proceed with this model, a necessary divergence is due at this point to present option pricing methodologies, based on scenario generation procedures.

4 Option pricing via scenario generation

The use of options on stocks is a fundamental means of hedging portfolios from market risk. The inclusion of options in a portfolio optimization framework naturally requires option pricing procedures fitting the optimization environment.

Before presenting option pricing schemes fitted into the portfolio optimization framework, a brief reference to fundamental concepts on option elements and strategies is made in the next section.

4.1 Fundamental options elements and strategies

A *call(put)* option on a stock gives the holder of the option the right to buy(sell) the stock by a certain date, T years from the day of purchase (option maturity), for a certain price K (option strike price).

From the definition of an option, one can directly derive the yield to the holder of the option as a function of the price S_T of the underlying stock (or any asset in general) at the option maturity T. For a holder of a "long" position on a call(put) option with strike price K, the option payoff at maturity is given by:

(35) $$\text{Put option payoff} = \max(K - S_T, 0)$$

(36) $$\text{Call option payoff} = \max(S_T - K, 0)$$

Of course, to compute the profit from exercising an option one must first compute the option's price and subtract it from the option's payoff function. A graphical representation of a put option's profit function, with strike price K=\$70 and option price \$7, is depicted in Figure 2.

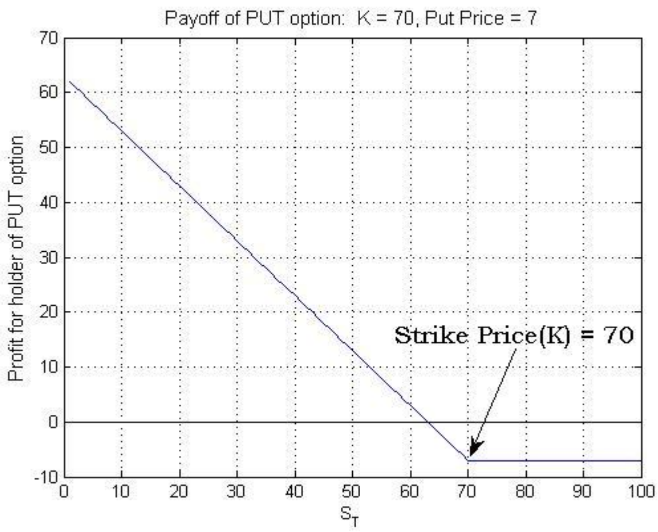

Figure 2 : Example of payoff function of a put option with strike price K= \$70 and option price P = \$7

A fundamental issue associated with options is the determination of their price. The standard option pricing formula on a non-dividend paying stock, dominating the financial industry for decades is the famous Black and Scholes (B-S) option pricing model[15], which involves the stock's volatility (σ), the time to option maturity (T), the market risk free rate (r_f), the strike price (K) and the stock's price (S_0) at the time of computation, as follows:

(37) $$C = S_0 N(d_1) - K e^{-r_f T} N(d_2)$$

(38) $$P = K e^{-r_f T} N(-d_2) - S_0 N(-d_1)$$

where

(39) $$d_1 = \frac{\ln(S_0 / K) + (r_f + \sigma^2 / 2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

An alternative detailed methodology on option pricing is presented in the next section.

4.1.1 *Hedging long positions with put options*

A very important aspect of the options functionality as a financial engineering tool is that of hedging. We are focusing on put options only, since this is the option category used in the context of the numerical studies presented in a later chapter.

Assume a portfolio consisting of a long position on a single stock with current price $S_0 = 70$ (prices implicitly assumed in \$). The simple put option hedging principle is to buy a put option on the stock. The overall portfolio profit, after exercising the put option at maturity, as a function of the stock price S_T at maturity time T, is expressed as:

(40) *Put option hedged portfolio profit*: $\max(K - S_T, 0) - P + (S_T - S_0)$

where P is the put option price. Three versions of this fundamental hedging strategy aree discerned, depending on the put option strike price, relative to the current stock price.

4.1.1.1 *Hedging with At the Money (ATM) Put options*

The put option strike price equals the stock's current price $K = S_0 = 70$.

The graph of the hedged portfolio profit as a function of the stock price (S_T) at maturity, is shown in Figure 3.

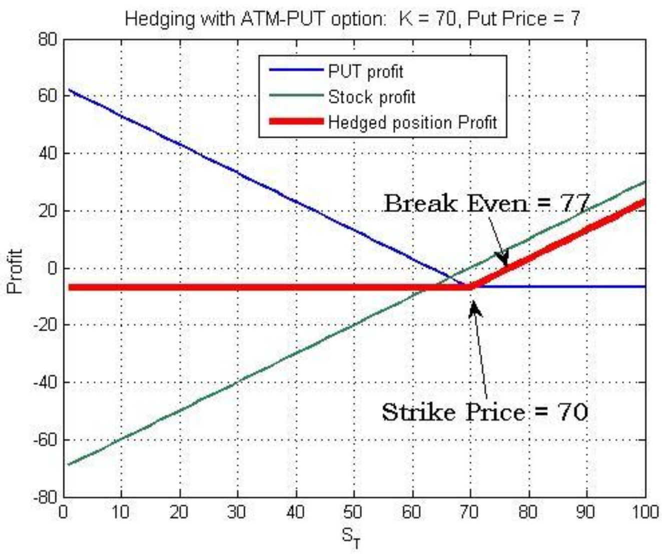

Figure 3: Hedging a long position on a stock with price S_0=70, with an ATM - Put option ($K = S_0$). Price option = 7

As can be seen by the red line in Figure 3, the overall portfolio profit is kept steady if the stock price falls lower than the strike price K=70, and follows the stock profit curve above the strike price. It becomes positive at the break even point $S_T = 77$.

4.1.1.2 *Hedging with Out of the Money (OTM) Put options*

The put option strike is lower than the stock's current price $K = 60 < S_0$. The graph of the hedged portfolio profit as a function of the stock price (S_T) at maturity, is shown in Figure 4.

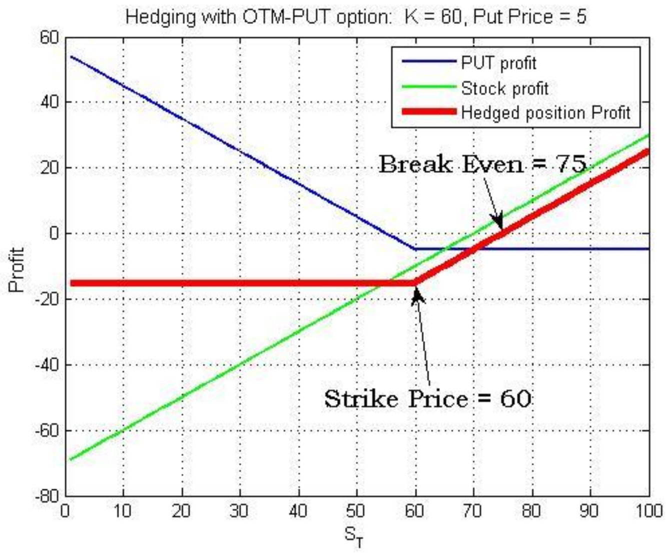

Figure 4: Hedging a long position on a stock with price S_0=70, with an ATM - Put option ($K = 60 < S_0$). Price option = 5

As can be seen by the red line in Figure 4, the overall portfolio profit is kept steady if the stock price falls lower than the strike price K=60, and follows the stock profit curve above the strike price. It becomes positive at the break even point $S_T = 75$. Compared to the ATM strategy, OTM hedging offers protection for a narrower range of the stock values ($S_T < 60$ as opposed to $S_T < 70$). The trade-off for this disadvantage, is the reduced option price (P=5), which allows for a lower break even price $S_T = 75$.

4.1.1.3 ***Hedging with In the Money (ITM) Put options***

The put option strike is higher than the stock's current price $K = 80 > S_0$. The graph of the hedged portfolio profit as a function of the stock price (S_T) at maturity, is shown in Figure 5

Figure 5: Hedging a long position on a stock with price S_0=70, with an ITM - Put option ($K = 80 > S_0$). Price option = 11

As can be seen by the red line in Figure 5, the overall portfolio profit is kept steady if the stock price falls lower than the strike price K=80, and follows the stock profit curve above the strike price. It becomes positive at the break even point $S_T = 75$. Compared to the ATM strategy, OTM hedging offers protection for a wider range of the stock values ($S_T < 80$ as opposed to $S_T < 70$). The trade-off for this advantage, is the increased option price (P=11), which result in a higher break even price $S_T = 82$.

The fundamental principle of the three versions of put option hedging strategies presented in this chapter is applied to a portfolio consisting of 20 stocks. The numerical results are presented in detail in chapter 1.

4.2 Pricing options on a scenario tree

The classical Black and Scholes (B-S) option pricing model[15] offers a convenient solution on option pricing, as long as the assumption that stock prices follow a geometric Brownian motion is adopted. It has been observed, though, that the B-S model can exhibit mispricing for deep out of the money (OTM) options, when the price distributions exhibit flat tails, which is the case in practice[16-[17]. Since, option pricing is obviously of extraordinary importance to the finance community, a huge volume of research has been conducted on the topic since the introduction of the seminal work of Black and Scholes. In this chapter, we choose to present an option pricing method which fits the scenario generation environment constructed for risk measure computation and portfolio optimization, in the previous chapters. The option pricing algorithm presented in this chapter shall be used subsequently to incorporate options into the portfolio optimization via stochastic programming. In the following sections of this chapter, the option pricing approach presented is adopted from the very recent work in [18].

For the sake of consistency, in the rest of this chapter we follow the notation introduced in section 2.1 for the scenario generation concept in order to present the scenario based option pricing procedure.

We consider an option on a portfolio asset (i) (stock) at current time t. The underlying asset price is $S_{i,t}$. Since discrete time is adopted in the scenario generation philosophy, we shall assume that the option maturity τ is placed at the next time epoch (e.g. one month from present time). According to the scenario generation procedure, the set of candidate prices considered for the underlying assets at time τ is given by (3) and repeated here for convenience

$$\left\{\tilde{S}_{i,n}\right\}_{n=1}^{N_S} = \left\{S_{i,t} R_{i,t-n}\right\}_{n=1}^{N_S} \quad , \quad 1 \le i \le N$$

This is a finite, hence countable, set. The discrete probability distribution $\{p_n\}_{n=1}^{N_S}$ for each of the underlying asset prices at time τ shall be referred to as the "physical" probabilities, and should satisfy (5) to be a legitimate probability measure.

There might be certain subtleties involving the theoretical rigidity of the distribution $\{p_n\}$. In a strict sense, the probability of a continuous random variable, such as a stock price, achieving a specific value should be equal to zero, since the event $\{S_{i,\tau} = \tilde{S}_{i,n}\}$ is a set of measure zero. However, assigning non-zero probabilities to such events can be justified if one thinks that stock prices are not exactly continuous random variables. Stock prices are rounded up to 2 decimal digits, i.e., the set of their possible values is indeed a countable set. Moreover, it is practically finite, since no stock price is higher than a few hundred dollars. In this sense, assigning non-zero probabilities to such events is legitimate.

Another practical way to circumvent this problem would be to accept the continuity of a stock return ratio $R_{i,n} = S_{i,t-n} \setminus S_{i,t-n-1}$ as a random variable but quantize its value set. For example, partitioning the return value set $[0,+\infty)$ into N_S subsets $([0,q_1),[q_1,q2),...,[q_{N_S-1},+\infty))$ and assigning discrete probabilities to each subset according to the frequency of occurrence observed from historical data, one can obtain a completely legitimate, and meaningful physical discrete probability distribution for the scenario set, as follows:

(41) $$P(\tilde{S}_{i,n}) = P\{R_{i,n} \in [q_n, q_{n+1})\} = p_n, \quad 1 \le n \le N_S$$

These discrete probabilities are guaranteed to be a legitimate probability measure, since the whole value set of the stock price is covered by the constructed partition. The detailed estimation procedure of the discrete set $\{p_n\}$ from historical data is not discussed further at this point.

From the option pricing theory we know that, to perform option pricing based on the discrete scenario set $\left\{\tilde{S}_{i,n}\right\}_{n=1}^{N_S}$, besides the physical probability measure, one needs to compute a risk-neutral probability measure on the same set, so as to impose the martingale property stating that the option price at current time should be equal to the expected option value at expiration, discounted by the risk free rate. In the next section the computation of the risk-neutral probability measure on the scenario set is presented in detail.

4.3 Risk neutral probability measure computation

A model of asset prices is arbitrage-free iff there exists a probability measure $\overline{P}$ (the risk-neutral measure) under which the discounted process of the asset prices is a martingale[18]. To derive a risk neutral probability measure $\overline{P} = \{\overline{p}_n\}$ on the scenario set $\left\{\tilde{S}_{i,n}\right\}_{n=1}^{N_S}$, consider an option on the i-th asset, with exercise price K and maturity τ equal to the scenario discrete time quantization step. Let r_f be the riskless rate applicable during the lifetime of the option. The martingale property for the discounted asset prices under the risk neutral probability measure $\overline{P}$ is expressed as

(42) $$E_{\overline{P}}[e^{-r_f \tau}\tilde{S}_{i,n} \mid S_{i,t}] = S_{i,t} \Leftrightarrow \sum_{n=1}^{N_S} \overline{p}_n \tilde{S}_{i,n} = e^{r_f \tau} S_{i,t} \ , 1 \le i \le N$$

For $\overline{P}$ to be a legitimate probability measure it must be that

(43) $$\sum_{n=1}^{N_S} \overline{p}_n = 1$$

The equivalence between the physical P and the risk neutral $\overline{P}$ probability measures poses the requirement of nonnegative probability masses in the scenario set, i.e.,

(44) $$\overline{p}_n > 0, \ 1 \le n \le N_S$$

Equations (43) - (44) are necessary conditions for the probability measure $\overline{P}$ to be a risk neutral measure for the scenario set. However, they are not sufficient, because the linear system representing the risk neutral measure usually involves less equations than unknowns. In order to achieve higher quality approximation to the discrete probability for the price scenarios, the size N_S of the scenario set is usually much larger than the number N of assets, rendering the linear system (42) - (44) underdetermined.

To circumvent this problem the authors in [18] resorted to a discrete time analogy of the procedure in [19], relating the physical and risk-neutral probability measures through a pricing kernel. Adjusting the pricing kernel to the power utility function of a representative market agent with coefficient of relative risk aversion γ, they ended up with the following expression for the risk neutral measure

(45) $$\overline{p}_n = \frac{e^{-\gamma r_n} p_n}{\sum_{n=1}^{N_S} e^{-\gamma r_n} p_n}, \ 1 \le n \le N_S$$

where $r_n = \ln(\tilde{S}_n / S_t)$. The relative risk aversion coefficient γ is estimated once, via fitting a number of computed option prices with corresponding market option prices at a date prior to the scenario present time t, and then used as is for subsequent discrete time epochs. To compensate for possible changes in the distribution of asset returns at subsequent discrete time epochs, they impose anew the risk neutral constraints at each time epoch via fitting the targeted risk neutral probability distribution to the "equilibrium" probabilities' distribution (45), in the least squares sense. At each stage the following minimization problem is solved

$$
(46)\qquad \begin{aligned}
\min_{\bar{p}_n} \quad & \sum_n \Big(\bar{p}_n - \frac{e^{-\gamma r_n} p_n}{\sum_{n=1}^{N_S} e^{-\gamma r_n} p_n}\Big)^2 \\
s.t. \quad & \sum_n \bar{p}_n \tilde{S}_n = e^{r_f \tau} S_t \qquad , \quad 1 \le n \le N_S \\
& \sum_n \bar{p}_n = 1 \\
& \bar{p}_n > 0
\end{aligned}
$$

Once the risk neutral discrete probability measure $\bar{P}$ is determined the pricing of options is derived in a straightforward manner because of the martingale property. Thus the price expressions for European put and call options with strike price K are, respectively:

$$
(47)\qquad P_t(S_t, K) = e^{-r_f \tau} \sum_{n=1}^{N_S} \bar{p}_n \max(K - \tilde{S}_n, 0)
$$

$$
(48)\qquad C_t(S_t, K) = e^{-r_f \tau} \sum_{n=1}^{N_S} \bar{p}_n \max(\tilde{S}_n - K, 0)
$$

5 Optimization of portfolio using options

In this chapter we apply the stochastic programming based portfolio optimization technique minimizing CVaR, along with the option pricing methods presented in the previous chapters to formulate a combined portfolio optimization and option pricing application. The presentation of the application model in this chapter is oriented as a guideline towards direct implementation in a stochastic program. The problem specifications are expressed appropriately so as to fit the jargon of stochastic programming (sets, parameters, variables, equations, etc.).

We assume that the portfolio initially consists of only cash M_0, which is entirely consumed to purchase stocks and options from the domestic market, i.e., there are no currency exchange rates involved. We adopt discrete time analysis. Each time epoch, referred to as "stage", corresponds to the planning horizon (e.g. month) of the portfolio manager, i.e., portfolio rebalancing occurs at the end of each stage. Present time is referred to as t. To describe the portfolio management operations we introduce the following notation:

Input Parameters

M_0: amount of cash at start of stage 0 (initialization of portfolio)

a: level of confidence for the CVaR risk measure

δ: proportional transaction cost for purchases and sales of assets

K_i: strike price of put option on asset i at start of stage t

$Put(S_t, K_i)$: price of put option on asset i at start of stage t

$S_{i,t}$: Price of asset i at start of stage t

Scenario dependent parameters

p_n: probability of scenario n

$\tilde{S}_{i,n}$: price of stock i under scenario n at end of stage t

Decision Variables

$w_{i,t}$: Position of stock i (in units of currency) at end of stage t

$x_{i,t}$: units of stock i purchased at end of stage t

$v_{i,t}$: units of stock i sold at end of stage t

$nP(S_t, K_i)$: units of put option on stock i, purchased at start of stage t and exercised at end of stage t.

Computed Parameters

Total value of the portfolio at stage t.

(49) $$V_t = \begin{cases} M_t & t = 0 \\ \sum_{i=1}^{N} w_{i,t-1} S_{i,t} & t > 0 \end{cases}$$

Auxiliary variables

$V_{t,n}$: total value of portfolio at end of stage t, under scenario n, after revision

L_n: portfolio loss at end of stage t, under scenario n, after revision

y_n: excess portfolio loss beyond VaR, under scenario n

ζ: VaR_α of portfolio losses at stage t

Equation (49) denotes that, initially the portfolio consists only of cash, which is totally consumed at the first stage, for purchase of assets (stocks and options). At subsequent stages cash is not part of the portfolio value.

Having defined the entities participating in the model we proceed with the equations (constraints) imposed for portfolio optimization via minimization of the computed portfolio CVaR measure.

Minimization metric:

(50) $$\min_{\mathbf{x},\zeta} F_\alpha(\zeta) = \zeta + (1-\alpha)^{-1}\sum_n p_n y_n$$

In (50), the functional $F_\alpha(\zeta)$ is the CVaR functional, presented in (33). This is the minimization metric.

Cash balance at portfolio initialization (t=0)

(51) $$M_0 = \underbrace{(1+\delta)\sum_{i=1}^{N} x_{i,0}S_{i,0}}_{\text{Cost of Purchased stocks at stage 0}} + \underbrace{\sum_{i=1}^{N} n_P(S_0,K_i)\cdot P(S_0,K_{i,0})}_{\text{Cost of Purchased options at stage 0}}$$

Equation (51) implies that all the initial cash is consumed for purchasing stocks and options on them.

Cash balance at stage t>0:

(52) $$\underbrace{\sum_{i=1}^{N} n_P(S_{t-1},K_{i,t-1})\cdot P(S_{t-1},K_{i,t-1})}_{\text{Yields from exercised options at stage t-1}} + \underbrace{(1-\delta)\sum_{i=1}^{N} v_{i,t}S_{i,t}}_{\text{Yields from stock sales at stage t}}$$
$$= \underbrace{(1+\delta)\sum_{i=1}^{N} x_{i,t}S_{i,t}}_{\text{Cost of stock purchases at stage t}} + \underbrace{\sum_{i=1}^{N} n_P(S_t,K_i)\cdot P(S_0,K_i)}_{\text{Cost of options purchases at stage t}}$$

Portfolio value under scenario n at stage t:

(53) $$V_{t.n} = \sum_{i=1}^{N} w_{i,t} \tilde{S}_{i,n} + \sum_{i=1}^{N} n_P(S_t, K_{i,t}) \cdot \max(K_{i,t} - \tilde{S}_{i,n}, 0)$$

In (53) the term $\tilde{S}_{i,n}$ is defined as in (3). It denotes the price of the i-th stock under scenario n at stage t. The first term in (53) represents the total value of the portfolio stocks under scenario n, while the second term is the total yield of the portfolio options under scenario n.

Portfolio percentage loss under scenario n at stage t:

(54) $$L_n = 1 - \frac{V_{t,n}}{V_{t-1,n}}$$

Constraints on the auxiliary variable y_n:

(55) $$0 \le y_n, \quad y_n \ge L_n - \zeta$$

Auxiliary variable y_n represents excess portfolio loss beyond VaR, under scenario n and should be nonnegative.

Constraints on the decision variable $w_{i,t}$

(56) $$w_{i,t} = w_{i,t-1} + x_{i,t} - v_{i,t}$$

(57) $$x_{i,t} \ge 0$$

(58) $$w_{i,t} \ge 0$$

(59) $$0 \le v_{i,t} \le w_{i,t-1}$$

Constraint (56) simply states that the amount of stocks held (for each asset i) at the portfolio at the end of stage t, after portfolio revision, should be equal to the amount held at stage t-1 plus the purchased stocks minus the sold stocks. Constraint (57) simply excludes any negative numbers for stock purchases to be derived from the optimization algorithm, since they do not make any physical sense. Constraint (58), implies that negative stock holdings are not allowed, i.e., short positions are not considered. Constraint (59) implies that, for each asset, the amount of stocks sold at the end of stage, t after portfolio revision, cannot exceed the amount already held in the portfolio at the start of stage t. A Summary of the constraint equations and optimization variable definition is shown in Table 1

$$\min_{\mathbf{x},\zeta} F_\alpha(\zeta) = \zeta + (1-\alpha)^{-1} \sum_n p_n y_n$$

$$M_0 = (1+\delta)\sum_{i=1}^{N} x_{i,0} S_{i,0} + \sum_{i=1}^{N} n_P(S_0, K_i) \cdot P(S_0, K_{i,0})$$

$$\sum_{i=1}^{N} n_P(S_{t-1}, K_{i,t-1}) \cdot P(S_{t-1}, K_{i,t-1}) + (1-\delta)\sum_{i=1}^{N} v_{i,t} S_{i,t}$$
$$= (1+\delta)\sum_{i=1}^{N} x_{i,t} S_{i,t} + \sum_{i=1}^{N} n_P(S_t, K_i) \cdot P(S_0, K_i)$$

$$V_{t.n} = \sum_{i=1}^{N} w_{i,t} \tilde{S}_{i,n} + \sum_{i=1}^{N} n_P(S_t, K_{i,t}) \cdot \max(K_{i,t} - \tilde{S}_{i,n}, 0)$$

$$L_n = 1 - \frac{V_{t,n}}{V_{t-1,n}}$$

$$0 \le y_n, \quad y_n \ge L_n - \zeta$$

$$w_{i,t} = w_{i,t-1} + x_{i,t} - v_{i,t}$$

$$x_{i,t} \ge 0, \quad w_{i,t} \ge 0$$

$$0 \le v_{i,t} \le w_{i,t-1}$$

Table 1: Equations for CVaR portfolio optimization backtesting

6 Numerical studies of portfolio optimization techniques

In this chapter, the performance of the analytically presented stochastic programming techniques for portfolio optimization via minimization of the CVaR metric, is assessed via numerical studies. The portfolio optimization techniques have been implemented in the form of computer programs, by means of software packages, such as Matlab and GAMS. Actual financial data time series have been utilized as input to stochastic programs. The results produced correspond to various portfolio optimization environments:

1. Totally unhedged portfolios, consisting of only stocks. No options on the portfolio stocks are considered to cover market risk.

2. Portfolios consisting of stocks and put options with strike prices equal to the stock prices (at the money options - ATM). The put options are chosen as a means of hedging against market risk. The option pricing is performed according to the scheme presented in chapter 4.

3. Portfolios consisting of stocks and put options with strike prices lower or higher than the stock prices (in the money (ITM) options, out of the money (OTM) options). The variations on the put options' strike prices are tested in an attempt to investigate the effect of the option price variation on the overall portfolio optimization performance.

For each of these environments, the performance of all algorithms was investigated by conducting backtesting ranging in time from January 2006 to June 2011.

6.1 Input Data collection

The candidate stocks for synthesizing the test portfolio were chosen to be the 20 largest in capitalization stocks from the S&P500 index, of the New York Stock Exchange, at the end of May 2011. For each stock a time series of monthly adjusted close prices were considered, ranging from January 1995 to June 2011. Adjusted close prices were chosen so as to incorporate the effects of splits and dividends. The stock prices time series were downloaded from the Yahoo web site (http://finance.yahoo.com/).

In all numerical studies the discrete time step (time distance between stages) was equal to one month. At each stage of the backtesting procedure, the scenario size was chosen to be $N_S = 100$, i.e., to form the candidate prices for each asset for the next stage, the asset's returns observed 100 months previous to current time were considered. In all numerical studies the confidence level for the CVaR function was kept at $\alpha = 0.99$.

To perform option pricing on the S&P500 stocks, the required time series of risk free interest rates were downloaded from the Federal Reserve web site "http://www.federalreserve.gov/ ".

6.1.1 *Descriptive statistics of assets monthly returns*

Before proceeding with the actual numerical studies on portfolio optimization, a flavor of the statistical properties of the monthly returns time series for the participating assets is given below. The first four order moments of all the selected S&P500 stock monthly returns for the time period January 1995 to June 2011 are summarized in Table 2.

	Mean	Standard Deviation	Kurtosis	Skewness	Minimum	Maximum
XOM	-0,0092	0,04926	0,91393	0,109448	-0,18902	0,131859
MSFT	-0,006	0,10045	3,80395	0,748525	-0,28956	0,523205
WMT	-0,0064	0,06771	1,24175	0,397122	-0,20922	0,262396
JNJ	-0,0078	0,0566	1,0218	0,405527	-0,14845	0,190955
PG	-0,0066	0,06666	24,0061	3,117401	-0,20095	0,548624
IBM	-0,0088	0,08302	1,60344	0,423384	-0,26133	0,292608
T	-0,0026	0,07378	0,93107	0,441619	-0,22661	0,230165
AAPL	-0,0063	0,16747	22,4966	3,24352	-0,31175	1,365683
JPM	-0,0033	0,10196	3,04229	1,112185	-0,2474	0,440471
PEP	-0,0067	0,06352	8,80024	1,551257	-0,16217	0,396767
CVX	-0,0088	0,05867	0,89201	0,373341	-0,19197	0,180209
CSCO	-0,0038	0,12255	3,65986	1,301617	-0,27995	0,580263
BAC	0,00534	0,13985	24,9966	3,631477	-0,42245	1,139571
WFC	-0,0058	0,09439	11,2137	1,894945	-0,28855	0,559276
KO	-0,0043	0,06683	2,10364	0,896275	-0,1821	0,236267
ORCL	-0,0053	0,13296	3,15259	0,840939	-0,39469	0,53244
INTC	-0,0018	0,12855	8,1165	1,839978	-0,25282	0,801608
PFE	-0,0055	0,06793	0,26768	0,473297	-0,15337	0,214286
HPQ	-0,0019	0,10829	2,49594	0,834755	-0,26147	0,470609
VZ	-0,0029	0,06996	1,84201	0,06199	-0,28195	0,265774

Table 2: Descriptive Statistics of stock returns

The leftmost column of Table 2 contains the Reuters codes for the participating S&P500 stocks. As can be observed, for most cases the kurtosis is far away from the value of 3, which characterizes a normal distribution. Although Table 2 does not exhibit output of detailed statistical tests on the return distribution of the involved assets, it can be readily deduced that the return distributions must exhibit flat tails. These numerical indications justify the motivation for use of the CVaR risk measure, as analyzed in more detail in previous chapters.

In the following sections the numerical study scenarios mentioned above are analytically presented.

6.2 Totally unhedged portfolio

In this environment the portfolio consisted of only stocks. No options were purchased for hedging against market risk. An initial cash amount M_0=\$10,000 was assumed. The cash was totally consumed for stock purchases at stage t=0. At subsequent stages (t>0) the implicit cash yield from stock selling, due to portfolio rebalancing at the end of each month, was again totally consumed for stock purchasing. At each stage t, the portfolio synthesis was determined via minimization of the portfolio loss CVaR and application of the relative constraints (cash balance etc), as in the algorithmic scheme presented analytically in chapter 5.

The portfolio returns at each stage are plotted in Figure 2. The returns of the S&P500 index, as well as the S&P500 adjusted close monthly time series is also shown as a reference. In Figure 6, both cases for transaction percentage cost of δ=0.35% and δ=0 (transaction cost - free) are shown. For this transaction cost level the difference in the portfolio monthly return time series of the CVaR based algorithm is barely discernible for most months.

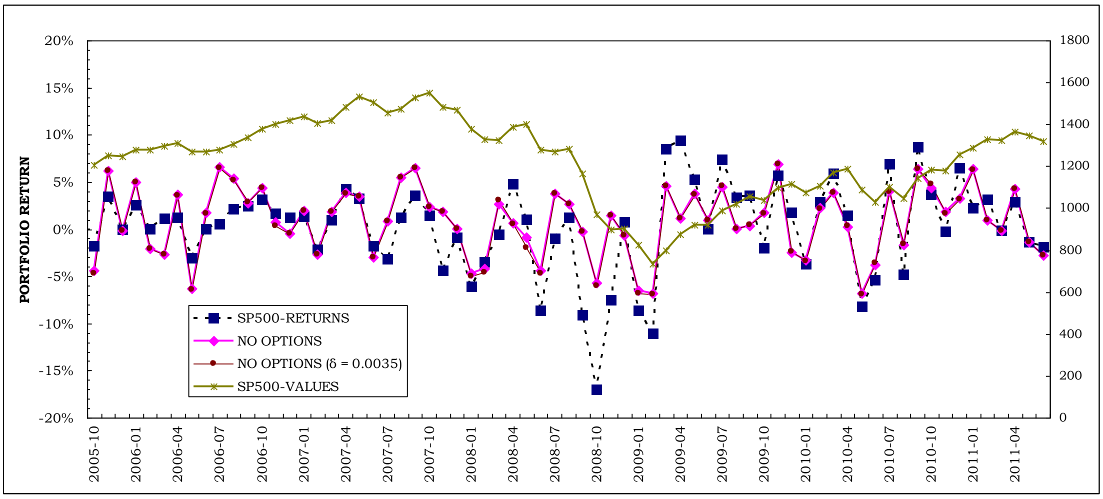

Figure 6: Portfolio returns backtest from 10/2005 to 6/2011. Totally unhedged portfolio. S&P500 monthly returns and S&P500 close values also shown for reference.

In general terms, Figure 6 shows that the performance of the CVaR portfolio optimization algorithm, in terms of monthly returns, closely follows that of the S&P500. This is quite reasonable, taking into account that the portfolio synthesis. It is interesting, however, to focus on the time interval around 2008-11, at the top of the credit crisis, when the S&P500 exhibited a very deep and steep plunge at a multiyear low. The returns of the CVaR based portfolio optimization algorithm seems to exhibit a noticeably more stable performance in terms of portfolio return, than the S&P500. For the months September 2008 to March 2009, the S&P500 returns were consistently lower than the CVaR algorithm by a margin of around 8%. The CVaR algorithm, however, did not respond as rapidly at the "recovery" period, starting April 2009 until September 2009. It appears that the CVaR based portfolio scheme represents a more risk averse portfolio management solution, with smaller variations in its return performance.

6.3 Hedging with ATM put options

In this environment, for each stock in the portfolio, a put option on the stock was purchased, with a strike price equal to the stock closing value at the start of each stage t, before portfolio rebalancing. The ATM options were purchased for hedging against market risk. An initial cash amount M_0=$10,000 was again assumed. The cash was totally consumed for stock and put option purchases at stage t=0. At subsequent stages (t>0) the implicit cash yield from stock selling, due to portfolio rebalancing at the end of each month, was totally consumed for stock purchasing. Moreover, the yields from exercising the options at the end of each stage (month) t, were also consumed for purchasing stocks and options at the start of the next stage t+1. At each stage t, the portfolio synthesis (stocks and options positions) was determined via minimization of the portfolio loss CVaR and application of the relative constraints (cash balance etc), as in the algorithmic scheme presented analytically in chapter 5.

For option pricing the algorithm presented in Chapter 4 was implemented. Due to lack of market prices information on put options for the portfolio stocks during the extended range of the backtesting period, the risk aversion coefficient γ was not determined via a least squares fit scheme to the market option prices, as described in Chapter 4. Empirical knowledge obtained from extended numerical studies on the subject[18], has shown that a value of $\gamma = 2$ is an appropriate rule of thumb for most cases. This was the value assigned to the risk aversion coefficient throughout the numerical studies involving put options on the portfolio stocks.

The portfolio returns at each stage of the backtesting time period are plotted in Figure 7. For comparison purposes the unhedged (no options) portfolio returns of the CVaR scheme are also presented in the figure. The returns of the S&P500 index, as well as the S&P500 adjusted close monthly time series, is also shown as a reference. For reasons of clarity, in Figure 7Figure 6, no transaction costs were considered (δ=0). It is understood that a higher transaction cost would somewhat deteriorate the performance of the CVaR based portfolio scheme, as with any conceivable scheme. However, this is not the most interesting performance merit.

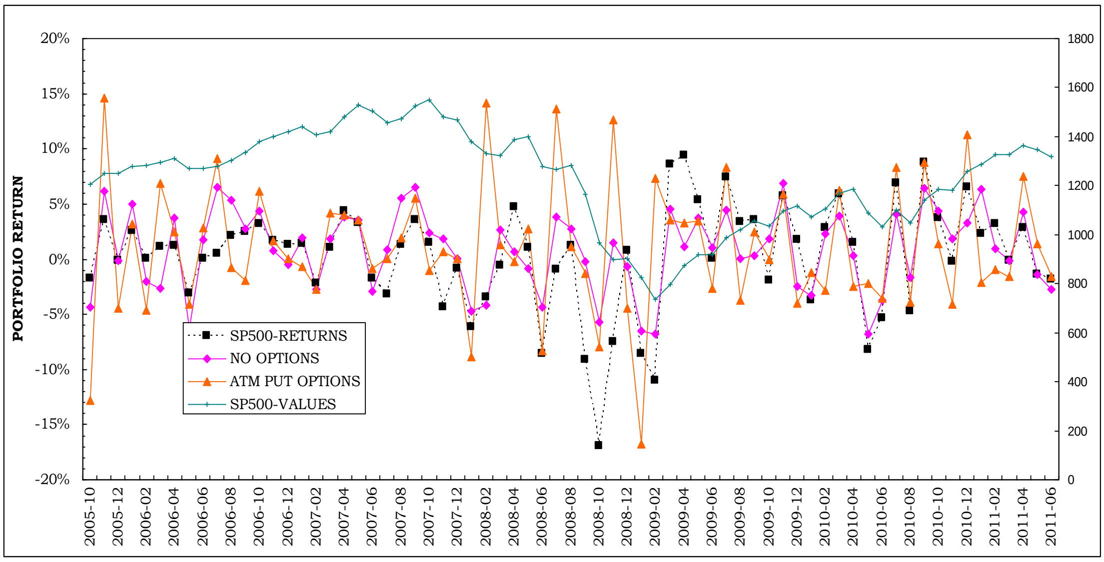

Figure 7: Portfolio returns backtest from 10/2005 to 6/2011. Portfolio hedged via ATM put options. Unhedged portfolio returns, S&P500 monthly returns and S&P500 close values also shown for reference.

As we observe in Figure 7, the ATM options scheme performance, in terms of monthly returns, is not far from that of the unhedged portfolio for most parts of the backtesting period. However, it is interesting to note several "spikes" in the performance of the ATM put option scheme, occurring at stages, when the S&P500 return time series recovers from plunges, as for example in months February 2008, July 2008, and November 2008. At these stages the ATM put option scheme exhibited returns surpassing those of the S&P500 and the unhedged portfolio scheme by more than 10 percentage points. However, a symmetrically opposite trend was observed at stages (months) when the S&P500 returns time series exhibited abrupt plunges, as for example in January 2009. For that month, the S&P500 returns were at a level of -8.57%, the unhedged CVaR portfolio returns were -6.48%, while the ATM put option scheme exhibited a return of -16.77%.

At stages when the S&P500 monthly returns' time series was ranging within the interval (-5%, 5%), both the unhedge CVaR scheme and the

ATM put option scheme were closely following the S&P500 performance, as expected, in view of the portfolio synthesis.

Focusing on the time interval from around 2008-11 until March 2009, at the heart of the credit crisis, the ATM put option CVaR scheme, compared to the unhedged scheme, exhibited more variation, being more responsive to steep spikes of the S&P500 but also to abrupt plunges.

6.4 Hedging with ITM and OTM put options

In this environment, for each stock in the portfolio, a put option on the stock was purchased, with a strike price either lower or higher than equal to the stock closing value at the start of each stage t, before portfolio rebalancing. In the first case the put option is referred to as "out of the money" (OTM), whereas in the second case as "in the money (ITM)". The put options were purchased for hedging against market risk. Other than the options' strike price, all the backtesting environment parameters for the case of ITM and OTM put options' portfolio were kept identical to the case of the ATM put options.

6.4.1 *Hedging with OTM put options*

For the case of OTM put options, the portfolio returns at each stage of the backtesting time period are plotted in Figure 8. At each stage t of the backtesting period, for each stock of the portfolio a put option was purchased with a strike price 5% lower than the close price at the end of the immediately previous stage (month). As a consequence, the option pricing algorithm was deriving a price for such an option lower than the corresponding ATM put option. Consequently, the optimal portfolio synthesis derived from the CVaR minimization algorithm would be differentiated with respect to the ATM put option case. For comparison purposes the returns of the ATM put options hedged portfolio are also presented in Figure 8. The returns of the S&P500 index, as well as the S&P500 adjusted close monthly time series, are also shown for reference. For reasons of clarity, in Figure 8Figure 6, no transaction costs were considered (δ=0).

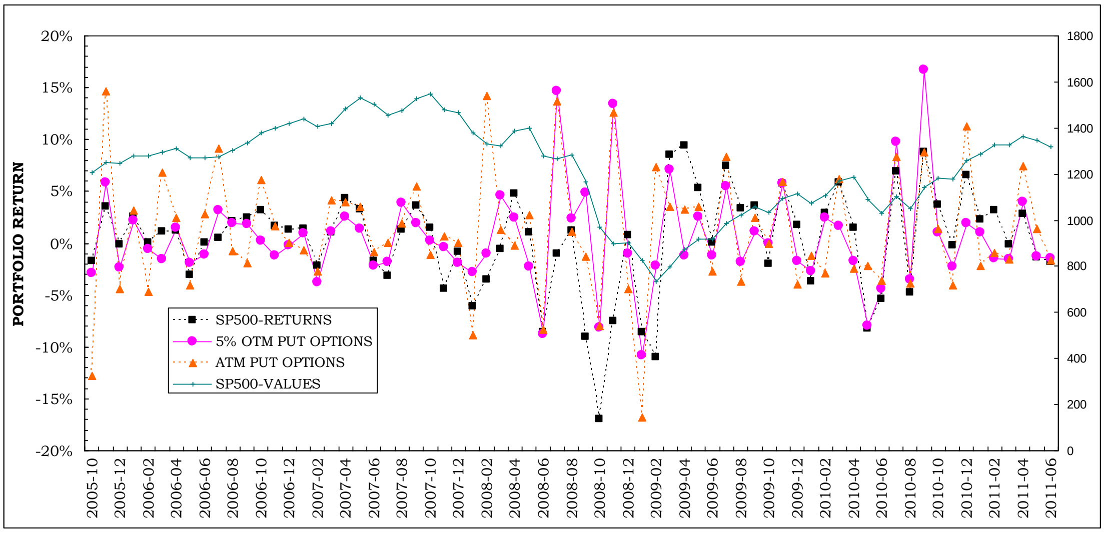

Figure 8: Portfolio returns backtest from 10/2005 to 6/2011. Portfolio hedged via OTM put options. ATM-put options hedged portfolio returns, S&P500 monthly returns and S&P500 close values also shown for reference

As we observe from Figure 8, in general, the 5%-OTM options hedged portfolio performance, in terms of monthly returns, is not far from that of ATM options hedged portfolio for most parts of the backtesting period. However, it is interesting to note that most of the "plunges" in the 5%-OTM options hedged portfolio returns are visibly mediated compared to those of ATM put option hedged portfolio, whereas most of the "spikes" are largely preserved. For example, for month July of 2008, the 5%-OTM option portfolio exhibits an even stronger spike in performance than the ATM option portfolio, at 14.7%, while ATM was at 13.63%, the unhedged portfolio was at 3.78% and the S&P500 returns were at -0.99%. Yet, for the month of October 2008, at the peak of the credit crisis, when the S&P500 returns were at -16.94%, the 5%-OTM hedged portfolio "preserved" a monthly return of -8.12%, very close to that of the ATM put options hedged portfolio.

Focusing on the time interval from around 2008-11 until March 2009, at the heart of the credit crisis, the 5%-OTM put option hedged portfolio, exhibited less deep plunges in performance than the ATM put option hedged portfolio, when the S&P500 was "sinking". However, it did not manage to follow the S&P500, and to a lesser extent the ATM put option hedged portfolio's performance, at the months immediately following the multiyear low of March 2009. For example, in April 2009 the S&P500 returns were at 9.39%, the unhedged portfolio were at 1.14%, the ATM options hedged at 3.29%, while the 5%-OTM put options hedged portfolio returns were at -1.22%.

6.4.2 *Hedging with ITM put options*

For the case of ITM put options, the portfolio returns at each stage of the backtesting time period are plotted in Figure 9. At each stage (t) of the backtesting period, for each stock of the portfolio a put option was purchased with a strike price 2% higher than the close price at the end of the immediately previous stage (month). As a consequence, the option pricing algorithm was deriving a price for such an option, higher than the corresponding ATM put option, leading to differentiated synthesis of the CVaR minimization algorithm output portfolio, with respect to the ATM put option hedged portfolio case. For comparison purposes the returns of the ATM put options hedged portfolio are also presented in Figure 9. The returns of the S&P500 index, as well as the S&P500 adjusted close monthly time series, are also shown for reference. For reasons of clarity, in Figure 9Figure 6, no transaction costs were considered (δ=0).

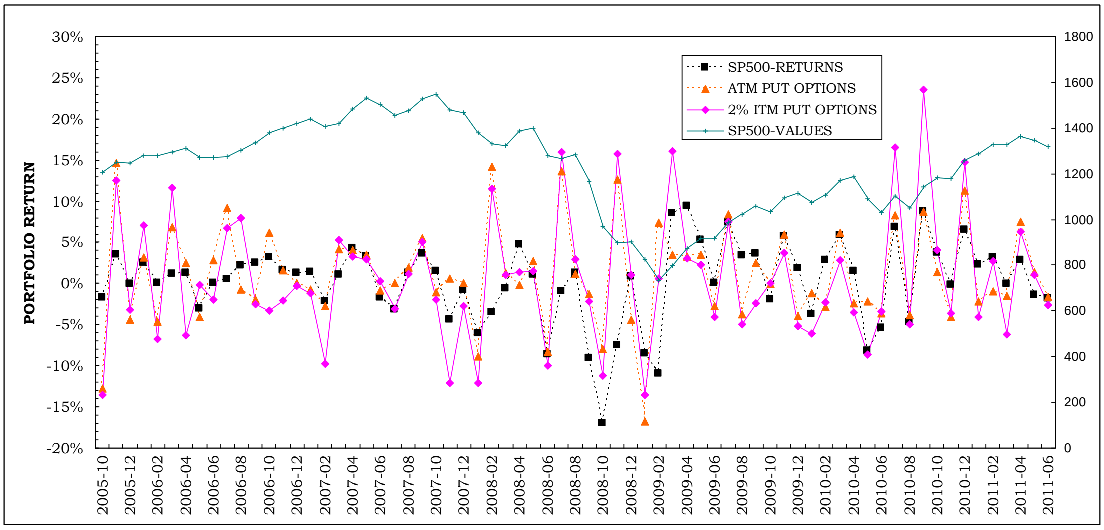

Figure 9: Portfolio returns backtest from 10/2005 to 6/2011. Portfolio hedged via ITM put options. ATM-put options hedged portfolio returns, S&P500 monthly returns and S&P500 close values also shown for reference.

As we observe from Figure 9, in general, the 2%-ITM options hedged portfolio performance, in terms of monthly returns, exhibits visibly higher variations from that of the ATM options hedged portfolio for significant parts of the backtesting period. There is a tendency to exaggerate in returns at stages (months) when the S&P500 exhibits local trend reversals to the upside. Conversely, there is a tendency to underperform at stages (months) when the S&P500 exhibits local trend reversals to the downside.

For example, for month July of 2008, the 2%-ITM put options hedged portfolio exhibits an even stronger spike in performance than all other schemes, at 15.96%, while 5%-OTM put options hedged portfolio was at 14.7%, ATM put options hedged portfolio was at 13.63%, the unhedged portfolio was at 3.78% and the S&P500 returns were at -0.99%. However, for the month of October 2008, at the peak of the credit crisis, when the S&P500 returns were at -16.94%, the 2%-ITM put options hedged portfolio exhibited the worst loss from all CVaR schemes, at -11.24%.

For the time interval from around 2008-11 until March 2009, at the heart of the credit crisis, the 2%-ITM put options hedged portfolio, exhibited deeper plunges in performance than the both the ATM put options hedged portfolio, when the S&P500 was "sinking" and steeper spikes when the S&P500 was spiking. At the months following the multiyear low (April- June 2008) it lacked in performance compared to the other put option hedged schemes.

For the overall backtesting period the 2%-ITM put options hedged portfolio achieved the highest monthly return of 23,55% at September of 2010, when the S&P500 returns were at 8.76%, the unhegded portfolio were at 6.41%, the ATM put options hedged portfolio at 8.78% and the 5%-OTM put options hedged portfolio returns were at 16.66%.

6.5 Performance comparison of for decision strategies

In this section a performance comparison of the portfolio optimization schemes presented so far is presented, in terms of the upside potential and downside risk ratio $\mathrm{UP}_{\mathrm{ratio}}$ metric[22]. Two different benchmark are used, namely the risk free rate of one-month T-bills and the S&P500 monthly returns. Denoting by r_t the realized portfolio returns and ρ_t the target (benchmark) returns, the $\mathrm{UP}_{\mathrm{ratio}}$ is computed as:

(60)
$$\mathrm{UP}_{\mathrm{ratio}} = \frac{\frac{1}{k}\sum_{t=1}^{K} \max\left[0, r_t - \rho_t\right]}{\sqrt{\frac{1}{k}\sum_{t=1}^{K} \left(\max\left[0, r_t - \rho_t\right]\right)^2}}$$

The statistics of the tested decision schemes are shown in Table 3.

	NO OPTIONS	ATM PUT OPTIONS	5% OTM PUT OPTIONS	2% ITM PUT OPTIONS
Standard Deviation	0,03659	0,05941	0,04696	0,07723
UP Ratio-FED	0,308111	0,418995	0,3524822	0,410442
UP Ratio-S&P500	0,882638	0,834333	0,7424078	0,712761

Table 3: Statistics of realized monthly returns

The standard deviation of all schemes is also shown in Table 3. We observe that in terms of the UPratio with target the monthly Fed rates, the ATM put options hedged portfolio scheme achieves the highest rank, with the 2% ITM scheme closely following. The no options scheme achieves the lowest rank among all schemes. The opossite is true when the target for the UPratio is the S&P500 monthly return performance. In this case the unhedged portfolio achieves the highest return with the ATM put options hedged portfolio closely following and the other two schemes clearly lagging.

7 Conclusions

In this document the problem of portfolio management was approached via the utilization of coherent risk measurement functions, namely, the Conditional Value at Risk. The theoretical framework of the risk measure based portfolio optimization was presented, along with the implementation aspects of the involved algorithmic schemes. Portfolio hedging techniques based on options were also examined within the same context. Stochastic programming techniques were employed to implement optimal portfolio synthesis. The performance of the pertinent schemes was investigated via numerical studies, utilizing a portfolio of 20 stocks from the S&P500 index. An extended time period ranging over the years 2006 - 2011 was considered for the numerical study, so as to include the period of the recent major credit crisis.

Although no target return constraints were imposed on the portfolio optimization algorithms tested, it was observed that, hedging schemes exhibited interesting performance during periods of extreme volatility. Based on the numerical studies results, especially during the high volatility periods of the credit crisis, we are motivated to assert that variation of certain parameters (e.g. strike price) of options employed as hedging tools, could certainly be a topic well worth further research efforts.

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9 APPENDIX

The listing of the source code of the GAMS (General Algebraic Modeling System) utilized to implement the stochastic programming models and produce the numerical results of chapter 1.

9.1 Code for hedging with put options

```
*=======================================================
$Title CVAR portfolio design including options
$Ontext
  The objective of this model is to find the optimum portfolio
  allocation minimizing CVAR
  Adjusted close time series is read from the GDX file "CVar_input.gdx"
$Offtext
*=======================================================
*--- Determine gdx file to pass data
$set output_to_matlab "'CVar_output.gdx' ";
sets
  assets          portfolio assets
  time       months' set in the form MO_monthstart * MO_monthend   ;
  alias (assets, i);
  alias (time, n);

*========PARAMETER DEFINITIONS ==================
*-- PARAMETERS READ FROM gdx FILE--------
PARAMETERS
      OptionFlag       Equals to1 if options included and 0 otherwise
  ReturnTargetFlag   equals to 1 if portfolio return target is set and 0 otherwise
      ReturnTarget             Portfolio return GE ReturnTarget
      Assetprices(i,n)         Price of asset(i) under senario n
    Portfprevious  Portfolio value after previous stage has been completed
  holdprevious(i)   Position in asset(i) after previous stage has been completed
      OptionPrice(i)           Price of Put options on assets
      OptionYieldSenario(i,n)  Yield of put option on asset i under senario n
      OptionsExercisedPrevious Total yields from options exercised at end of previous stage
(needed for Cash Balance equation)
      CashInitial              Position in cash at initial stage
      backteststage            number of current backtest stage(month)
      ;
```

```
*-- SENARIO DEPENDENT and other PARAMETERS --------
PARAMETERS
        prob(n)                 Scenario probability
        P0(i)     Price of asset(i) at beginning of stage process (most recent value of stage window)
        senario(i,n)            Price of asset(i) under senario n
        denom(i)                auxilliary parameter
        ;
*======== END OF PARAMETER DEFINITIONS ===========================
PARAMETERS
        alpha         Confidence level
        modelStat      lala
        solveStat      lala        ;
*------------------------------------------------
*------------------------------------------------
*   READ SETS, AND PARAMETERS FROM GDX FILE
$gdxin CVar_input
$load assets time Assetprices holdprevious CashInitial backteststage
$load   alpha  OptionFlag ReturnTargetFlag  ReturnTarget
$load   OptionsExercisedPrevious OptionPrice OptionYieldSenario

$gdxin
*---------------------------------
*=========================================
*=========================================
*-- Initialize parameters

*-- Compute physical probabilities ----------------
         prob(n) = 1.0 / card(n);
*-- Compute asset prices before portfolio revision.  ----------------
*--  These are the close prices of the immediately previous month and
*--  are considered as the spot prices of the underlying assets at start of new month

*         loop(n$(ord(n) eq card(n)), P0(i) = AssetPrices(i,n));
         loop(n$(ord(n) eq (card(n)-1)), P0(i) = AssetPrices(i,n));

*=========================================

*-- Initial portfolio value ----------------
   Portfprevious = SUM(i, holdprevious(i) * P0(i)) + CashInitial$(backteststage=1);
*                 + OptionsExercisedPrevious$(OptionFlag=1) + CashInitial$(backteststage=1);
*=====================================================================
```

```
*-- Compute price senarios for each asset  ----------------
     senario(i,n)$(ord(n) > 1) = P0(i) *  AssetPrices(i,n)/AssetPrices(i,n-1);
     senario(i,n)$(ord(n) eq 1) = P0(i) ;

*=============================================
*=============================================
*-- END OF PARAMETER INITIALIZATION ===

*======== DECISION VARIABLES DEFINITIONS ============================

*POSITIVE VARIABLES
VARIABLES
*-- Definitions in actual quantities (e.g. 1000 stocks)  - (also covers equation (2l))
*         buy(i)          Purchased units of asset(i) during current stage-- (x_i)
*         sell(i)         Sold units of asset(i) during current stage --  (v_i)
        metavoli(i)     Units of asset(i) bought or sold at current stage;

POSITIVE VARIABLES
        hold(i)           Held units of asset(i) at end of current stage -- (w_i)
        OptionHold(i)    Held option contracts of asset(i) (exercised) at end of current stage
        VaR               Value-at-Risk  (z in paper) -- Partly covers equation (2j)
        VaRDev(n)        Excess Loss beyond VaR under senario n (y_n) ;

*======== AUXILIARY VARIABLES DEFINITIONS ============================
VARIABLES
       PosChange       Total value of buy and sell positions within each stage
       Portf(n)        Total Value of revised portfolio under senario n --  (V_n)
       Loss(n)         Portfolio (percentage) loss under senario n -- (L_n)
       CVar              Objective function value   (in the paper: F = z+(1-a)(^-1)*sum_n(p(n)*y(n)) ;

*======== EQUATIONS DEFINITIONS ============================
EQUATIONS
       ChangePositionValue Net worth of position change should be equal to initial cash at first
stage and zero afterwards
        CashBalanceDef      Cash balance condition at each stage -- Equation (2b)
        PortfVal(n)         Value of portfolio at end of stage under senario n -- Equation (2d)
        PortfLoss(n)        Portfolio Loss under senario n -- Equation (2h)
        ReturnCon           Portfolio return constraint -- Equation (2i)
        VaRDevCon(n)        Excess portfolio loss beyond VaR -- Equation (2j)
        HoldAsset(i)        Balance condition for eash asset --Equation (2k)
        ChangeAsset(i)      Cannot change position to negative (no short sell asset (i)) --equivalent
to Equation (2m)
```

```
    ObjDefCVaR          Objective function definition for CVaR minimization-- Equation (2a);

ChangePositionValue..  PosChange =E= SUM(i, metavoli(i)*P0(i))
                       +SUM(i, OptionHold(i)*OptionPrice(i) )$(OptionFlag=1);
CashBalanceDef..     PosChange =E= CashInitial$(backteststage = 1)+
                       OptionsExercisedPrevious$((OptionFlag=1) and (backteststage > 1));
PortfVal(n)..        Portf(n) =E= SUM(i, hold(i) * senario(i,n)+
                       OptionHold(i)*OptionYieldSenario(i,n)$(OptionFlag=1));
PortfLoss(n)..       Loss(n) =E=  1 - Portf(n)/Portfprevious;
ReturnCon$(ReturnTargetFlag > 0) ..    SUM(n, prob(n) * (-Loss(n))) =G= ReturnTarget;
VaRDevCon(n) ..      VaRDev(n) =G= Loss(n) - VaR;
HoldAsset(i)..       hold(i) =E=  holdprevious(i) + metavoli(i);
ChangeAsset(i)..     holdprevious(i) + metavoli(i) =G= 0;

*-- Equation (2a)
ObjDefCVaR ..        CVar =E= VaR + SUM(n, prob(n) * VaRDev(n)) / (1 - alpha);
*======== END OF EQUATIONS DEFINITIONS ===========================
*=== MODEL DEFINITION   ===========================
MODEL MinCVaRPortfolio /ChangePositionValue, CashBalanceDef,PortfVal,PortfLoss,
                ReturnCon,VaRDevCon, HoldAsset,ChangeAsset,ObjDefCVaR/;
*=== END OF MODEL DEFINITION   ===========================
*------------------------------------------------------------
    SOLVE MinCVaRPortfolio MINIMIZING CVar USING LP;
*------------------------------------------------------------
    modelStat = MinCVaRPortfolio.MODELSTAT;
    solveStat = MinCVaRPortfolio.SOLVESTAT;
*================================================
execute_unload %output_to_matlab%;
*$call "gdx2xls CVar_output"
```

9.2 Code for unhedged portfolio

```
$Title CVAR portfolio design not including options
$Ontext
  The objective of this model is to find the optimum portfolio
  allocation minimizing CVAR
  Adjusted close time series is read from the GDX file "CVar_input.gdx"
$Offtext

*====================================================
*--- Determine gdx file to pass data
$set output_to_matlab "'CVar_output.gdx' ";
*=======================================================
*=======================================================
sets
  assets          portfolio assets
  time            months' set in the form MO_monthstart * MO_monthend
  ;
  alias (assets, i);
  alias (time, n);

*======== PARAMETER DEFINITIONS ===========================
*-- PARAMETERS READ FROM gdx FILE--------
PARAMETERS
      ComissionRate          comission rate for sales and purchases
      ReturnTargetFlag        equals to 1 if portfolio return target is set and 0 otherwise
      ReturnTarget           Portfolio return GE ReturnTarget
      Assetprices(i,n)       Price of asset(i) under senario n
      Portfprevious          Portfolio value after previous stage has been completed
      holdprevious(i)        Position in asset(i) after previous stage has been completed
      CashInitial            Position in cash at initial stage
      backteststage          number of current backtest stage(month)
      ;
*  ComissionRate = 0.0;

*-- SENARIO DEPENDENT and other PARAMETERS --------
PARAMETERS
      prob(n)                Scenario probability
      P0(i)                  Price of asset(i) at beginning of stage process (most recent value of
stage window)
      senario(i,n)           Price of asset(i) under senario n
```

```
    denom(i)            auxilliary parameter
    ;

*======== END OF PARAMETER DEFINITIONS ===========================

PARAMETERS
    alpha       Confidence level
    modelStat    lala
    solveStat    lala
    ;

*-----------------------------------------------
*-----------------------------------------------
*  READ SETS, AND PARAMETERS FROM GDX FILE
$gdxin CVar_input
$load assets time Assetprices holdprevious CashInitial backteststage
$load  alpha  ReturnTargetFlag  ReturnTarget
$load ComissionRate

$gdxin
*=========================================
*-- Initialize parameters

*-- Compute physical probabilities ----------------
    prob(n) = 1.0 / card(n);
*-- Compute asset prices before portfolio revision.  ----------------
*--  These are the close prices of the immediately previous month and
*--  are considered as the spot prices of the underlying assets at start of new month

    loop(n$(ord(n) eq (card(n)-1)), P0(i) = AssetPrices(i,n));

*=========================================
*-- Initial portfolio value ----------------
  Portfprevious = SUM(i, holdprevious(i) * P0(i)) + CashInitial$(backteststage=1);
*=================================================================



*-- Compute price senarios for each asset  ----------------
   senario(i,n)$(ord(n) > 1) = P0(i) *  AssetPrices(i,n)/AssetPrices(i,n-1);
   senario(i,n)$(ord(n) eq 1) = P0(i) ;
*-- END OF PARAMETER INITIALIZATION ===
```

```
*======== DECISION VARIABLES DEFINITIONS ============================
POSITIVE VARIABLES
        hold(i)           Held units of asset(i) at end of current stage -- (w_i)
        buy(i)          Purchased units of asset(i) during current stage-- (x_i)
        sell(i)        Sold units of asset(i) during current stage --  (v_i)
        VaR              Value-at-Risk  (z in paper) -- Partly covers equation (2j)
        VaRDev(n)        Excess Loss beyond VaR under senario n (y_n) ;

*======== AUXILIARY VARIABLES DEFINITIONS ============================
VARIABLES
       PosChange       Total value of buy and sell positions within each stage
       Portf(n)        Total Value of revised portfolio under senario n --  (V_n)
       Loss(n)         Portfolio (percentage) loss under senario n -- (L_n)
       CVar              Objective function value   (in the paper: F = z+(1-a)(^-1)*sum_n(p(n)*y(n)) ;

*======== EQUATIONS DEFINITIONS ============================
EQUATIONS
        ChangePositionValue Net worth of position change should be equal to initial cash at first
stage and zero afterwards
        CashBalanceDef      Cash balance condition at each stage -- Equation (2b)
        PortfVal(n)         Value of portfolio at end of stage under senario n -- Equation (2d)
        PortfLoss(n)        Portfolio Loss under senario n -- Equation (2h)
        ReturnCon           Portfolio return constraint -- Equation (2i)
        VaRDevCon(n)        Excess portfolio loss beyond VaR -- Equation (2j)
        HoldAsset(i)        Balance condition for eash asset --Equation (2k)
        ChangeAsset(i)      Cannot change position to negative (no short sell asset (i)) --equivalent
to Equation (2m)
        SellAsset(i)        Cannot sell more units of asset(i) than we have at beginning of stage -
        ObjDefCVaR          Objective function definition for CVaR minimization-- Equation (2a);



ChangePositionValue..  PosChange =E= SUM(i, buy(i)*P0(i)*(1+ComissionRate)
                             -sell(i)*P0(i)*(1- ComissionRate));


*-- Equation (2b)
CashBalanceDef..     PosChange =E= CashInitial$(backteststage = 1) + 0$(backteststage > 1);



*-- Equation (2d)
PortfVal(n)..        Portf(n) =E= SUM(i, hold(i) * senario(i,n));


*-- Equation (2h)
```

```
PortfLoss(n)..        Loss(n) =E=  1 - Portf(n)/Portfprevious;

*-- Equation (2i)
ReturnCon$(ReturnTargetFlag > 0) ..    SUM(n, prob(n) * (-Loss(n))) =G= ReturnTarget;

*-- Equation (2j)
VaRDevCon(n) ..       VaRDev(n) =G= Loss(n) - VaR;

*-- Equation (2k)
HoldAsset(i)..        hold(i) =E=  holdprevious(i) + buy(i) - sell(i);

*-- Equation (2m)
SellAsset(i)..        sell(i) =L= holdprevious(i);
ChangeAsset(i)..      holdprevious(i) + buy(i) - sell(i) =G= 0;

*-- Equation (2a)
ObjDefCVaR ..         CVar =E= VaR + SUM(n, prob(n) * VaRDev(n)) / (1 - alpha);
*======== END OF EQUATIONS DEFINITIONS ===========================

*=== MODEL DEFINITION   ===========================
MODEL MinCVaRPortfolio /ChangePositionValue, CashBalanceDef,PortfVal,PortfLoss,
                  ReturnCon,VaRDevCon, HoldAsset,ChangeAsset,
                  SellAsset, ObjDefCVaR/;

*=== END OF MODEL DEFINITION   ===========================
*---------------------------------------------------------
      SOLVE MinCVaRPortfolio MINIMIZING CVar USING LP;
*---------------------------------------------------------
      modelStat = MinCVaRPortfolio.MODELSTAT;
      solveStat = MinCVaRPortfolio.SOLVESTAT;
*==============================================
execute_unload %output_to_matlab%;

*$call "gdx2xls CVar_output"
```