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FUZZY CONTROL CHARTS

By

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ΟΙΚΟΝΟΜΙΚΟ ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ

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ΑΣΑΦΗ ΔΙΑΓΡΑΜΜΑΤΑ ΕΛΕΓΧΟΥ

Βαλεντίνη Πάρι Κυριακίδου

ΔΙΑΤΡΙΒΗ

Που υποβλήθηκε στο Τμήμα Στατιστικής

του Οικονομικού Πανεπιστημίου Αθηνών

ως μέρος των απαιτήσεων για την απόκτηση

Μεταπτυχιακού Διπλώματος Ειδίκευσης στη Στατιστική

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To my parents Paris and Maria,
and to my lovely grandparents.

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VITA

Kyriakidou Valentini was born in Cyprus in 1986. She graduated from the high school Archiepiskopou Makariou C in Nicosia in 2004. Also, she got a degree in Mathematics from the University of Patras in 2008, and a master's degree in Statistics from the Athens University of Economics and Business in 2011. In the graduate program she produced a thesis on "Fuzzy Control Charts".

ABSTRACT

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FUZZY CONTROL CHARTS

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Statistical Quality Control is the most popular application of statistical methods and is firmly established methodology with many practical applications. Statistical Process Control is the most important field of SQC where its main tools are control charts, first proposed by Shewhart in 1924, in order to monitor and examine the production process. The traditional control charts dealing with precise data. There are many situations where data are uncertain, vague or imprecise. Fuzzy set theory, first proposed by Zadeh in 1965, is probably the most appropriate tool to handle with this uncertainty, providing mathematical techniques in order to deal with imprecise concepts and problems. Using linguistic data (intermediate levels), first proposed by Wang & Raz in 1990, to describe the product quality can provide more information than the binary classification used in traditional control charts. Also, fuzzy multivariate control chart is an alternative control chart for handling linguistic observations, when more than one quality characteristics monitoring, simultaneously. Many investigators proposed several procedures for the construction of control charts when data are vague or imprecise, and concluded that, using fuzzy control charts can provide a more flexible and informative evaluation of the considered process. The purpose of the thesis is to review the basic techniques used to handle the uncertain information and also, to present the various fuzzy control charts proposed by several researchers.

ΠΕΡΙΛΗΨΗ

Βαλεντίνη Κυριακίδου

ΑΣΑΦΗ ΔΙΑΓΡΑΜΜΑΤΑ ΕΛΕΓΧΟΥ

Μάιος 2011

Ο Στατιστικός Έλεγχος Ποιότητας, είναι η πιο δημοφιλής εφαρμογή των στατιστικών μεθόδων με αποδεδειγμένα πολλές πρακτικές εφαρμογές. Η Στατιστική Διαδικασία Ελέγχου, είναι ο πιο σημαντικός τομέας του Στατιστικού Ελέγχου Ποιότητας με κύρια εργαλεία, τα διαγράμματα ελέγχου. Τα διαγράμματα ελέγχου, προτάθηκαν για πρώτη φορά από τον Shewhart το 1924, προκειμένου να παρακολουθούν και να εξετάζουν τη διαδικασία παραγωγής. Τα παραδοσιακά διαγράμματα ελέγχου, ασχολούνται με ακριβή δεδομένα. Όμως, υπάρχουν αρκετές περιπτώσεις, όπου τα δεδομένα είναι ασαφής, αβέβαια είτε αόριστα. Η ασαφής θεωρία ελέγχου, η οποία προτάθηκε για πρώτη φορά από τον Zadeh το 1965, είναι ίσως το καταλληλότερο εργαλείο για το χειρισμό αυτής της αβεβαιότητας, παρέχοντας μαθηματικές τεχνικές για την αντιμετώπιση ασαφών εννοιών και προβλημάτων. Οι Wang & Raz το 1990, για την περιγραφή της ποιότητας των προϊόντων, χρησιμοποίησαν γλωσσικά δεδομένα (ενδιάμεσα επίπεδα) τα οποία παρέχουν περισσότερη πληροφορία από τη δυαδική ταξινόμηση η οποία χρησιμοποιείται από τα παραδοσιακά διαγράμματα ελέγχου. Επίσης, τα πολυμεταβλητά ασαφή διαγράμματα ελέγχου, είναι μια εναλλακτική λύση για το χειρισμό των γλωσσικών παρατηρήσεων, όταν περισσότερα από ένα ποιοτικά χαρακτηριστικά παρακολουθούνται ταυτόχρονα. Πολλοί ερευνητές πρότειναν διάφορες διαδικασίες για την κατασκευή των διαγραμμάτων ελέγχου, όταν τα δεδομένα είναι ασαφή ή ανακριβή και κατέληξαν στο συμπέρασμα ότι, χρησιμοποιώντας ασαφή διαγράμματα ελέγχου επιτυγχάνεται μια πιο ευέλικτη και κατατοπιστική αξιολόγηση των υπό εξέταση διαδικασιών. Ο σκοπός αυτής της εργασίας είναι η αναθεώρηση των βασικών τεχνικών που χρησιμοποιούνται για το χειρισμό της αβεβαιότητας

καθώς επίσης και η παρουσίαση των διάφορων ασαφών διαγραμμάτων ελέγχου, τα οποία προτάθηκαν από τους διάφορους ερευνητές.

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CHAPTER 1

INTRODUCTION

As the needs of society increases, so the science is involving, and new statistical methods come to the forefront. Those statistical methods have many applications, such as in biostatistics (biometrics, clinical trials, epidemiology), in engineering statistics (probabilistic design, process and quality control), in social statistics (crime statistics, econometrics, population), in spatial statistics (cartography, geostatistics, environmental statistics), etc.

Statistical Quality Control (SQC), is the most popular application of statistical methods, and is firmly established methodology with many practical applications. The most important field of SQC is Statistical Process Control (SPC), where its main tools are control charts first proposed by Shewhart in the 1920s. The control charts deal with items that classifying either in a set or out of a set, such as “conforming” or “nonconforming”, in order to decide if an inspected item belongs in process control or not.

On the other hand, in many practical and realistic situations, it is difficult to classify inspected items strictly as “conforming” or “nonconforming”, and should be determined the degree to which is conforming or nonconforming. For example, if the temperature of water at $100°C$ degree is expressed as “hot”, so for water at $95°C$ or $80°C$ the expression of “not hot” is not wrong or right in this meaning.

So, the theory of fuzzy sets, first proposed by Zadeh in his papers in 1965, suggested by using values between wrong [0] and right [1] values. Also, fuzzy sets, provides useful tools for dealing with many problems related to the lack of precision in statistical data and imprecisely defined quality requirements.

Wang and Raz in the 1980s were fist dealt with the fuzzy data and proposed the construction of fuzzy control charts using linguistics data, such as "perfect", "good", "medium", "poor", and "bad", (intermediate levels in order to express the assessments on the evaluation of the products).

Since then, various procedures have been proposed for monitoring processes in which the data are fuzzy and many fuzzy control charts are constructed. Some important field of fuzzy logic and fuzzy control charts, are pointed out in the papers of Laviolette et al. (1995), Barrett & Woodall (1997), and Hryniewicz (2008), and also, in the paper of Woodall et al. (1997) presented a review of fuzzy control charts based on categorical data.

This thesis is organized in the following order. We will first present the Statistical Process Control (SPC), its basic concepts, as well as the main control charts in Chapter 2. Later, Chapter 3 contains the fuzzy set theory, the definitions for fuzzy random variables and the types of some fuzzy numbers. Fuzzy control charts based on linguistic data are presented in Chapter 4. In Chapter 5 several approaches for the construction of fuzzy control charts using any defuzzification methods or not, are presented analytically, and also, fuzzy multivariate control charts are presented in Chapter 6. Finally, conclusions are given in Chapter 7.

CHAPTER 2

STATISTICAL PROCESS CONTROL AND CONTROL CHARTS

2.1. Statistical Process Control (SPC)

The concept of Statistical Process Control (SPC) has established as the most efficient tool for on – line quality control in mass production system. SPC is a powerful collection of problem – solving tools, and also a methodology for monitoring a process in order to identify special causes of variation and to signal the requirement for corrective action when necessary. SPC is widely employed throughout industry and is a proven technique for improving quality and productivity.

The major tools of SPC are histogram, check sheet, Pareto chart, cause – and – effect diagram, defect concentration diagram, scatter diagram and control chart. The most common SPC technique is the statistical control chart, first developed by Shewhart during the 1920s, which has become a vital tool for professionals who seek to improve the quality of their products.

2.2. Control Charts

The popularity of control charts is because control charts are a proven technique for improving productivity by monitoring and examining production process. Also, they are effective in defect prevention having the ability to detect process shifts and to identify abnormal conditions in a production process. Additionally, control charts prevent unnecessary process adjustment and provide diagnostic information of many production problems and often reduce losses and bring substantial improvements in production quality. This is achieved by repeating the Phase I and Phase II. Furthermore, control charts provide information about process capability, and finally, it should be noted, that control charts are constructed with data collected from the process.

The general form of control charts consists of a centre line (CL), and two control lines referred to as the lower control limit (LCL) and the upper control limit (UCL) respectively. The CL represents an estimate of the process level, while the two control lines, LCL and UCL, denote the boundaries of normal variability and are specified such that the majority of the observations (and especially the 99.74% of them), lie within their bounded range when the process is under control. Points plotted on the chart represent samples drawn from the process.

In the case, where all points falling between control limits, indicates that there are no abnormal conditions in the production process, so require no action, and the process is said to be in statistical control. On the other hand, if an item is displayed out of control limits, indicates that some assignable or chance causes were present when the sample was drawn, and suggests the need for corrective action. This is achieved using the Phase II, and then repeating Phase I.

Various types of control charts have been developed in industry for controlling different types of quality characteristics. The basic principles of development and design of various types of control charts as Montgomery (2007) noted, have presented below.

2.3. Basic Principles

There are two distinct phases of control charting, the Phase I and the Phase II.

2.3.1. Phase I and Phase II

The purpose of Phase I is to evaluate the stability of the process, using a retrospective analysis of process data in order to construct trial limits. Also, the purpose of Phase I is to find and remove any outliers, measurement errors or data entry errors, with assignable causes, and estimate the in – control values of the process parameters. In designing a control chart, we must specify an appropriate sample size to use and the frequency of sampling. In order to establish reliable control limits, the number of samples to be taken is $m = 5$ each with size $n = 25$. It should be noted that, a larger sample size results in narrow control limits, as well, a smaller sample size makes the control limits wider.

On the other hand, the Phase II is a control phase. The main purpose of Phase II is to monitor the on – line data to quickly detect any shift in the process parameter from the baseline estimated in Phase I.

2.3.2. Statistical hypothesis testing

The statistical hypothesis testing is a useful tool in analyzing the performance of a control chart, and also, is an assumption about a population parameter. The best way to determine whether a statistical hypothesis is true would be to examine the entire population. Since that is often impractical, researchers typically examine a random sample from the population. If sample data are not consistent with the statistical hypothesis, the hypothesis is rejected. There are two types of statistical hypothesis:

Null hypothesis $(H_0) \rightarrow$ is usually the hypothesis that sample observations result purely from chance.
Alternative hypothesis $(H_1) \rightarrow$ is the hypothesis that sample observations are influenced by some non – random cause.

Using the above statistical hypothesis testing, two types of decision errors can result. These are the type I error and the type II error.

2.3.3. Errors in Control charts

The type I error occurs when the researcher reject the null hypothesis (H_0) when it is true. This means that the type I error is the result of concluding that a process is out – of – control when it is actually in – control. Therefore, the probability of type I error is: $P(type\,\mathrm{I}\,error) = a =$ significance level.

Also, the type II error occurs when the researcher fails to reject the null hypothesis (H_0) when it is false. This means that the type II error is the result of concluding that a process is in – control when it is actually out – of – control. Therefore, the probability of type II error is: $P(type\,\mathrm{II}\,error) = b =$ power of the test = the probability of non – detection.

In order not to have frequent false alarms, suppose that we have 3 – sigma (3σ) control charts. Thus we achieved a very small price of $a = 0.0027$. So we have small price of probability of type I error, and therefore less false alarms.

2.3.4. Curves of Control Charts

In order to evaluate the ability and the performance of the control charts, it is necessary to use one of the following curves ARL or OC, which are measures of ability and performance.

2.3.4.1. ARL – Curve

The Average Run Length (ARL), is the average number of points that must be plotted before a point indicates an out – of – control condition. The value of ARL depends on type I error. As we have seen above, the smaller the value of $P(type\,\mathrm{I}\,error) = a$, the higher the value of ARL $\left[ARL = \frac{1}{a}\right]$. Furthermore, the ARL is often used as a criterion to compare competing methods in Phase II.

2.3.4.2. OC – Curve

The Operating Characteristic Curve (OC), is a measure of the ability of a control chart to detect the changes in process parameters. This means that is a graph that constructed in order to show how changes in the sample size (n), affect the probability of making a type II error $[P(TYPE \text{ II } error) = b]$.

2.3.5. Process Capability Index

The process capability is also another important sense in SPC, which represents the performance of a process when it is in a statistical control. A process capability index $\left(C_p\right)$, uses both the process variability and the process specifications to examine the variability in process characteristics as well as whether the process is capable of producing products which conforms to specifications.

2.4. Types of Control Charts

Depending on the number, the form and interpretation of data collected, control charts classified into two general types, referred to as univariate and multivatiate.

2.4.1. Univariate Control Charts

The univariate type of control charts consists of Variable and Attribute Control Charts, and is a graphical display of one quality characteristic.

2.4.1.1. Variable Control Charts

When dealing with a quality characteristic can be expressed in terms of numerical measurement (is measured on a numerical scale) such as dimension, weight, volume, length, width, temperature, etc, is called variable. Usually, variable quality characteristics are normally distributed and monitored using Shewhart control charts for variables. The most common types of variable control charts are as follows:

$\overline{X} -$ Control Chart

- Control chart for means
- Used to monitor the process average or mean quality level
- Denoted the variability between samples

$R -$ Control Chart

- Control chart for the range
- Used to monitor the process variability
- Denotes the variability within samples
- Used for small sample size (n)

$S -$ Control Chart

- Control chart for the standard deviation
- Used to monitor the process variability
- Denotes the variability within samples
- Used for big sample size (n)

$MR -$ Control Chart

- Moving Range control chart
- The moving range is the absolute value of the difference between successive observations
- Used to monitor variable data for which it is impractical to use rational subgroups $(n = 1)$

$CUSUM -$ Control Chart

- Cumulative Sum Control Chart
- Used for monitoring process variability
- Used when information from all previous samples need to be used for controlling the process

- Points plotted on tabular CUSUM chart are the sum of two consecutive elements
- More effective in detecting small changes in the process mean compared to the other charts
- Greater sensitivity for detecting shifts or trends for individual data over the traditional Shewhart charts, because history is taking account

$MA -$ Control Chart

- Moving Average Control Chart
- On this chart, each point plotted is the average of the last k data values

$EWMA -$ Control Chart

- Exponentially Weighted Moving Average Control Chart
- Is very effective in detecting small shifts in process mean or variance, especially when the sample size is unity
- On this chart, each point plotted is a weighted average of all data values up until that point, where the more recent data is given more weight that the older data

2.4.1.2. Attribute Control Charts

When the quality characteristics cannot be represented in numerical form, such as characteristics for appearance, softness, color, etc, then control charts for attributes are used. In attribute control charts, product units are either classified as “conforming” or “non – conforming”, “good” or “bad” depending upon whether or not they meet specifications. The most common types of attribute control charts are as follows:

$P -$ Control Chart

- Control chart for the fraction (proportion) non – conforming items
- Used with binomial distributed data
- Used for controlling the ratio of defective products
- Tracks the proportion of nonconformities per sample

$nP -$ Control Chart

- Control chart for the number of non – conforming items per sample
- Counts the number of defectives in a sample

$C -$ Control Chart

- Control chart for the total number of non – conformities
- Used Poisson distributed data
- Used for controlling the number of defects per inspection unit of products

$U -$ Control Chart

- Control chart for the average number of non – conformities per unit
- The non – conformities in samples of constant size is well modeled by Poisson distribution
- Is designed for counting defects per sample when the sample size varies for each inspection, and used for controlling the mean number of defects of a product

2.4.2. Multivariate Control Charts

So far, we have seen process monitoring and control charts dealing with one quality characteristic. In the case we have to deal with several related characteristics, univariate control charts are inefficient and the use of multivariate control charts is required. Multivariate type of control charts is a graphical display of a statistic that summarizes or represents more than one quality characteristic. This means that the quality of the product is a function of many characteristics.

Assuming that there are p – quality characteristics (variables), computed separate univariate control chart for each variable and then combined these separate univariate statistics into a single control statistic and plot it on a control chart. This process seems to work well when the number of quality characteristics is less than 10.

First who worked with multivariate quality control was Hotteling in 1947. The T^2 distribution is used to develop the control chart, when the F distribution is used for finding the upper control limit. The lower control limit is set to be zero. The most common types of multivariate control charts are:

$Hotteling\ T^2 -$ Control Chart

- Using for monitoring the mean vector of the process
- Is insensitive to small and moderate shifts in the mean vector

$MEWMA -$ Control Chart

- Multivariate $EWMA$ control chart, extension of univariate $EWMA$ chart

$MCUSUM -$ Control Chart

- Multivariate $CUSUM$ control chart, extension of univariate $CUSUM$ chart

In order to reduce the dimensionality of the variable space, it is necessary to use projection methods like Principal Components Analysis or Partial Least Squares.

Multivariate process control is one of the most rapidly developing sections of statistical process control, having the advantage that can both handle process variables and product quality variables.

Montgomery (2007), Stoumbos et al. (2000), and Bersimis et al. (2007 a, b) presented analytically the principal notions of multivariate control charts, and also the main types of these charts.

Control charts are among the most important and widely used tools in statistics. Their applications have now moved far beyond manufacturing into engineering, environmental science, biology, genetics, epidemiology, medicine, finance, and even law enforcement and athletics.

CHAPTER 3

FUZZY SET THEORY AND FUZZY RANDOM VARIABLES

3.1. Fuzzy set theory

In several areas of science, there are many situations where researchers have to deal with data that are vague or imprecise. The most significant sources of uncertainty are randomness and incomplete or imprecise information. A worthwhile tool for expressing this uncertainty is fuzzy set theory.

Fuzzy set theory proposed a mathematical technique for dealing with imprecise concepts and imprecise problems that have many possible solutions. Also, fuzzy set is a mathematical model of vague qualitative or quantitative data, frequently generated by mean of the natural language. The model is based on the generalization of the classical concepts of set and its membership (or characteristic) function.

Zadeh (1965) first introduced the notion of fuzzy sets, as an extension of the classical notion of sets.

In classical set theory, the membership function of elements in a set is assessed in binary terms according to a bivalent condition. Consider the set A and the indicator function I_A which identifies if an element either belongs or does not belong to the set A.

Where,

$$I_A = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$.

In general, a fuzzy set is characterized by a membership function which assigns to each object a grade of membership between zero and one. The major contribution of fuzzy set theory is its capability to represent vague data.

3.1.1. Zadeh's definition of fuzzy sets

As Zadeh (1965) defined in his paper for fuzzy sets, the definition of fuzzy set is presented as: *"Let X be a space of points, with a generic element of X denoted by x. Thus $X = \{x\}$. A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ which associates with each point in X a real number in the interval $[0,1]$, with the value of $\mu_A(x)$ at x representing the «grade of membership» of x in A. Thus, the nearer the value of $\mu_A(x)$ to unity, the higher the grade of membership of x in A."*

So, the form of the membership function is:

$$\mu: A \to [0,1]$$

When $\mu_A(x) = 0$, means that x is not a member of the fuzzy set A. When $\mu_A(x) = 1$, means that x is fully a member of the fuzzy set A. And when $0 < \mu_A(x) < 1$, means that the fuzzy number x belongs to the fuzzy set A only partially.

3.1.2. Definitions and Theorems of fuzzy sets

The fuzzy set theory, allows mathematical operators to apply to the fuzzy domain. Several definitions involving fuzzy sets are following presented:

I. A is *empty* if and only if $\mu_A(x) = 0$ identically

II. A, B are *equal*, $(A = B)$, if and only if $\mu_A(x) = \mu_B(x)$ for all $x \in X$

III. The *complement* of A: $\mu_{A'}(x) = 1 - \mu_A(x)$.

Where, A' is the complement of A

IV. *Containment*: A is contained in B if and only if $\mu_A(x) \leq \mu_B(x)$:

$$A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$$

V. *Union*: The union of two fuzzy sets A and B with respective membership functions $\mu_A(x)$ and $\mu_B(x)$ is :

$$C = A \cup B \Longrightarrow \mu_C(x) = \max[\mu_A(x), \mu_B(x)]$$

{Property: $A \cup (B \cup C) = (A \cup B) \cup C$}

VI. *Intersection*: The intersection of two fuzzy sets A and B with respective membership functions $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set

$$C = A \cap B \Longrightarrow \mu_C(x) = \min[\mu_A(x), \mu_B(x)]$$

Also, some algebric operations on fuzzy sets are:

I. *Algebric Product*: $\mu_{AB}(x) = \mu_A(x)\mu_B(x)$

II. *Algebric Sum*: $\mu_{A+B}(x) = \mu_A(x) + \mu_B(x)$

III. *Absolute Difference*: $\mu_{|A-B|}(x) = |\mu_A(x) - \mu_B(x)|$

IV. Let A, B, Λ be arbitrary fuzzy sets. The *convex combination* of A, B and Λ, $(A, B;\ \Lambda)$, in terms of membership functions is:

$$\mu_{(A,B;\ \Lambda)}(x) = \mu_\Lambda(x)\mu_A(x) + [1 - \mu_\Lambda(x)]\mu_B(x)$$

$\left\{Property: \mu_{AB}(x) \leq \mu_{(A,B;\ \Lambda)}(x) \leq \mu_{A+B}(x)\right\}$

V. *Composition*: $\mu_{A \circ B}(x, y) = sup_v \min[\mu_A(x, v), \mu_B(v, y)]$

Where, x, y, v real numbers

VI. *Convexity*: A fuzzy set A is convex if and only if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_A(x_1), \mu_A(x_2)]$

Where,

$x_1, x_2 \in X$ (X is a real Euclidean space E^n)

$\lambda \in [0, 1]$

Theorem 1: If A and B are convex, so is their intersection.

VII. *Boundedness*: A fuzzy set A is bounded if and only if the sets $\Gamma_a = \{x | \mu_A(x) \geq a\}$ are bounded for all $a > 0$; that is, for every $a > 0$ there exists a finite $R(a)$ such that $\| x \| \leq R(a)$ for all x in Γ_a.

Theorem 2: If A is a convex fuzzy set, then its core is a convex set

Theorem 3: Let A and B ne bounded convex fuzzy sets in E^n, with maximal grades M_A and M_B respectively.

Where, $M_A = sup_x\big(\mu_A(x)\big)$ and $M_B = sup_x\big(\mu_B(x)\big)$. Let M be the maximal grade for the intersection $A \cap B$. Where, $M = sup_x(\min[\mu_A(x), \mu_B(x)])$.

Then $D = 1 - M$. Where, D is the degree of separability of A and B.

3.1.3. a – Cut

One particularly useful class of subsets comprises the elements of a fuzzy set with membership values larger than a given cutoff a. Especially, for every $a \in [0,1]$, a given fuzzy set A yields a crisp set $A^a = \{x \in X | A(x) \geq a\}$, which is called an a – cut of A. Thus, a – cut, is a non – fuzzy set which comprises all elements whose membership degrees are greater than or equal to a. Below presented the shapes of triangular fuzzy number and trapezoidal fuzzy number with a – cuts.

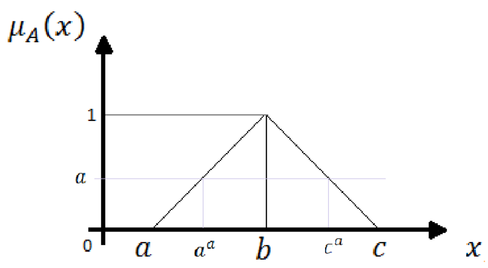

Figure 3.1. Triangular Fuzzy Number with α – cut

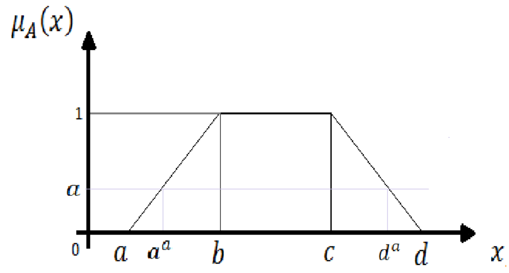

Figure 3.2. Trapezoidal Fuzzy Number with α – cut

3.2. Fuzzy random variables

Fuzzy random variable is a particular kind of fuzzy set. In addition, fuzzy random variables are random variables whose values are not real numbers but fuzzy numbers, and may be used to describe and characterize situations where we have to deal with statistical data that are imprecise.

Many researchers have been dealt with fuzzy random variables and different approaches have been developed. As Gil et al. (2006), Shapiro (2009) and Beer (2010), reviewed in their papers, the most widely considered definitions for fuzzy random variables were introduced first by Kwakernaak (1978 and 1979) and then by Puri & Ralescu (1986).

3.2.1. Kwakernaak definition of fuzzy random variables

Kwakernaak (1978 and 1979), considered that fuzzy random variables have been considered in the setting of a random experiment to model a fuzzy perception of a mechanism associating a real value with each experimental outcome.

Definition: A fuzzy random variable ξ defined on a probability space $(\Omega, \mathcal{F}, P)$ $\{$where Ω is the sample space, $\mathcal{F}$ is $\sigma -$ algebra of subsets of Ω, the set of all fuzzy numbers and P is a probability measure on $\Omega\}$, is characterized by a map $X: \Omega \to S$ such that $\omega \xrightarrow{X} X_\omega$, where S is a collection of all piecewise continuous functions $\mathcal{R} \to [0, 1]$. Each element of S is a membership function of fuzzy number. The map X is required to satisfy the following properties:

1. For each $\mu \in (0, 1]$, both ${U_\mu}^*$ and ${U_\mu}^{**}$ defined by:
   ${U_\mu}^* = \inf\{x \in \mathcal{R} | X_\omega(x) \geq \mu\}$
   ${U_\mu}^{**} = \sup\{x \in \mathcal{R} | X_\omega(x) \geq \mu\}$
   Are finite real valued random variables on $(\Omega, \mathcal{F}, P)$, and have finite mathematical expectations.

2. For each $\omega \in \Omega$ and each $\mu \in (0, 1]$,
   $X_\omega\left({U_\mu}^*(\omega)\right) \geq a$ and $X_\omega\left({U_\mu}^{**}(\omega)\right) \geq a$
   Eventually, a fuzzy random variable ξ is defined as a fuzzy set $\xi = \left(\widetilde{\mathcal{X}}, X\right)$, where $\widetilde{\mathcal{X}}$ is the set of possible originals of the fuzzy random variable ξ.

3.2.2. Puri &Ralescu definition of fuzzy random variables

Puri & Ralescu (1986), considered that fuzzy random variables have been considered in the setting of a random experiment to model an essentially fuzzy – valued mechanism, that is, a mechanism associating a fuzzy value with each experimental outcome.

Definition: A fuzzy random variable ξ is a function $\xi: \Omega \to \mathcal{F}_c(\mathcal{R})$ (a collection of all normalized fuzzy numbers whose $a-$ level sets are compact convex subset of $\mathcal{R}$ – the set of fuzzy subsets), such that:

$\{(\omega, x) | x \in \xi_a(\omega)\} \in \mathcal{F} \times B$ for every $a \in [0, 1]$

Where, $\xi_a: \Omega \to K_c(\mathcal{R})$ (the class of all nonempty compact convex subsets of $\mathcal{R}$), is a random set defined by $\xi_a(\omega) = \{x \in \mathcal{R} | \xi_\omega(x) \geq a\}$ and B denotes the collection of Borel subsets of $\mathcal{R}$. That is, in order to make fuzzy random variable ξ mathematically meaningful, Puri & Ralescu (1986), impose the hypothesis of measurability on the random set ξ_a associated with ξ.

3.3. Fuzzy numbers

Fuzzy numbers are an extension of real numbers, whose values are only vaguely defined. A fuzzy number is a special case of a convex fuzzy set and may assume different real values, with each of which a degree of acceptability is associated. Furthermore, calculations with fuzzy numbers allow the incorporation of uncertainty on parameters, properties, geometry, initial conditions, etc.

<u>Definition:</u> A fuzzy number is a fuzzy set with domain ℝ, the real numbers, that is normal, has bounded support and whose a – cuts are closed intervals for positive a.

Depending the nature and shape of membership function the fuzzy number can be classified in different forms, such as triangular fuzzy numbers, trapezoidal fuzzy numbers, LR – fuzzy numbers, etc.

3.3.1. Fuzzy triangular number

A fuzzy set A with the following membership function:

$$\mu_A(x) = \begin{cases} 0, & for\ x \leq b-a \\ 1-\frac{b-x}{a}, & for\ b-a < x < b \\ 1, & for\ x = b \\ 1-\frac{x-b}{c}, & for\ b < x < b+c \\ 0, & for\ x \geq b+c \end{cases} = \begin{cases} 1-\frac{b-x}{a}, & b-a \leq x < b,\ a > 0 \\ 1-\frac{x-b}{c}, & b \leq x \leq b+c,\ b > 0 \\ 0, & otherwise \end{cases}$$

is called *triangular fuzzy number (TFN)*, if there exists a triplet of real numbers $a, b, c \in \mathbb{R}$ with $a < b < c$, and denoted by $A = (a, b, c)$.

Its membership function, $\mu_A(x)$, also given by the following shape:

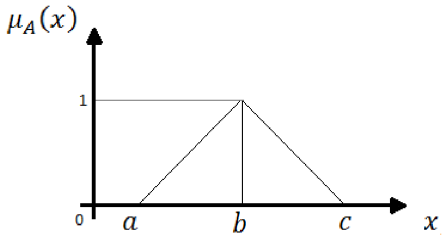

Figure 3.3. Fuzzy triangular number

A triangular fuzzy number A is said to be normalized if $\mu_A(x) = 1$.

Where,

$\mu_A(x) \in [0,1] \rightarrow$ The membership grade (or height)

$a \rightarrow$ The left hand spread

$c \rightarrow$ The right hand spread

$b \rightarrow$ The mean value

3.3.2. Fuzzy trapezoidal number

Also, a fuzzy set A with the following membership function:

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \frac{x-d}{c-d}, & c \le x \le d \\ 0, & otherwise \end{cases}$$

is called *trapezoidal fuzzy number*, if $a, b, c, d \in \mathbb{R}$ with $a < b < c < d$, and denoted by $A = (a, b, c, d)$.

Its membership function, $\mu_A(x)$, also given by the following shape:

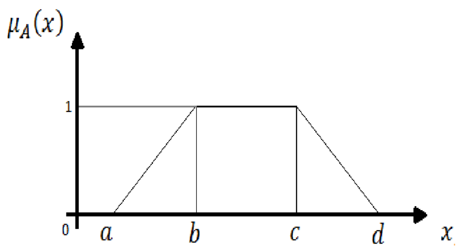

Figure 3.4. Fuzzy trapezoidal number

In the special case, when $b = c$, we have the *triangular fuzzy number* $A = (a, b, d)$.

3.3.3. LR – Fuzzy number

The LR – fuzzy number is a unimodal fuzzy number on $\mathbb{R}$ that can be described in terms of two shape functions. The left hand shape function L and the right shape function R.

The LR – fuzzy number can be described as $u = (m, l, r)_{LR}$ with the form as:

$$u(x) = \begin{cases} L\left(\frac{m-x}{l}\right), for\ x \leq m \\ R\left(\frac{x-m}{r}\right), for\ x > m \end{cases}$$

Where,
$L: \mathbb{R}^+ \to [0,1]$ and $R: \mathbb{R}^+ \to [0,1]$ are non – increasing functions with $L(0) = R(0) = 1$.
$m \to$ The central point of u
$l > 0 \to$ The left spread of u
$r \to$ The right spread of u

Also, a unimodal fuzzy set has its maximum value at a unique value m of the domain X. Especially, $u(x) = 1 \Leftrightarrow x = m$ and graphically,

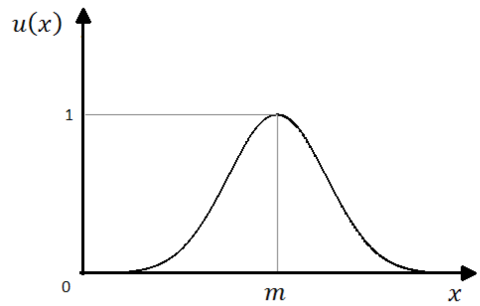

Figure 3.5. LR – Fuzzy number

3.4. Transformation methods

In order to obtain the representative value of a fuzzy set, various defuzzification methods are proposed. Defuzzification is a process that maps a fuzzy set to a crisp set. Some of these transformation methods presented as follow:

3.4.1. Fuzzy Mode ($\boldsymbol{f_{mode}}$)

$f_{mode} = \{x|\mu_A(x) = 1\},\ \forall x \in A$

The value of the base variable, x, where the membership function is equal to 1

3.4.2. $\boldsymbol{a}$ − level Fuzzy Midrange ($\boldsymbol{f_{mr}(a)}$)

$f_{mr}(a) = \frac{1}{2}(a_a + c_a)$

The average of the endpoints of an $a -$ level cut.

Where,

$A_a = \{x|\mu_A(x) \geq a\}$, the $a -$ level cut of all A

a_a, c_a, the endpoints of A_a

$a_a = \min[A_a]$

$c_a = \max[A_a]$

3.4.3. Fuzzy Median ($\boldsymbol{f_{med}}$)

Is the point which divides the area under the membership function into two equal regions, satisfying the equation:

$\int_a^{f_{med}} \mu_A(x)dx = \int_{f_{med}}^{c} \mu_A(x)\, dx = \frac{1}{2}\int_a^c \mu_A(x)dx$

Where,

a, c: The endpoints in the base variable, x, of the fuzzy set A

3.4.4. Fuzzy Average (f_{avg})

$$f_{avg} = Av(x;A) = \frac{\int_{x=0}^{1} x\mu_A(x)dx}{\int_{x=0}^{1} \mu_A(x)dx}$$

It should be pointed out, that there is no theoretical basis supporting any one specifically, and the selection among the transformation methods should be mainly based on the ease of computation, on the user's preference, or on the quality manager.

3.5. Applications of fuzzy sets

The fuzzy set theory and related branches are widely applied in the models of optimal control, decision – making under uncertainty, processing vague econometric or demographic data, behavioral studies and methods of artificial intelligence. Also, fuzzy sets can be applied in sociology, political science, and anthropology, as well as in any field of inquiry dealing with complex patterns of causation.

CHAPTER 4

FUZZY CONTROL CHARTS BASED ON LINGUISTIC DATA

4.1. Introduction

The binary classification into conforming or non – conforming might not be appropriate in many situations where product quality does not change abruptly from satisfactory to worthless and there might be a number of intermediate levels. These intermediate levels may be expressed in the form of linguistic variables, whose values are words or phrases in some language, such as “perfect”, “good”, “medium”, “poor”, and “bad”.

Using intermediate levels (linguistic data) to describe the product quality can provide more information than the binary classification used in control charts by attributes. The binary classification may result weaker detectability of process shifts or other abnormal conditions.

In order to retain the standard format of control charts and to facilitate the plotting of observations on the chart, it is necessary to convert the fuzzy sets associated with linguistic values into scalars referred to as representative values. This was done using one of the following transformation methods; by using the fuzzy mode (f_{mode}) , the α – level fuzzy midrange $\left(f_{mr}(a)\right)$, the fuzzy median (f_{med}), and the fuzzy average $\left(f_{avg}\right)$. The conversion of fuzzy sets into its representative values also retain the ambiguity and vagueness in natural languages, improves the expressive ability of assurance inspectors and constitute a more realistic approach to process control. It should be pointed out that the development of membership function it can be done with several methods; based on statistical data, polls, different beliefs, and other.

A sample consists of several observations selected for inspection. Each observation in a sample of linguistic data is a linguistic variable, (L_i), associated with a fuzzy set, (F_i), defined on the base variable, (X), and described by a known membership function, $(\mu_i(x))$. These linguistic values need to be combined to yield a single value for the sample. This combination may be done either before or after the conversion of fuzzy sets associated with linguistic terms into their representative values.

Several researchers have dealt with the construction of fuzzy control charts where the data is in the form of linguistic terms. Some of these approaches are presented in detail below.

4.2. Wang & Raz Approach

Wang and Raz (1990) were the first who dealt with the construction of fuzzy control charts, where the data are presented in the form of linguistic terms. They proposed two approaches for the construction of attribute control charts, using the fuzzy set theory, in order to monitor quality data given in the form of linguistic values.

In order to retain the standard format of control charts (UCL, CL, LCL), it is necessary to convert the fuzzy sets (associated with linguistic data) into their representative values, using one of the following transformation methods:

1. fuzzy mode: f_{mode}
2. α – level fuzzy midrange: $f_{mr}(\alpha)$
3. fuzzy median: f_{med}
4. fuzzy average: f_{avg}

Firstly, in order to yield a single representative value (M_j) for a sample (of linguistic values), a combination of those individual linguistic values is used, before or after their conversion into representative values.

4.2.1. Before the conversion

They proposed to add the corresponding fuzzy set for each linguistic value in a sample and then to divide by the number of them. The result (mean of the fuzzy sets : MF_s) is also a fuzzy set. The membership function of MF_s is:

$$\mu_s(x_s) = \max_{(x_1k_1+\cdots+x_tk_t)/_n}\{\min[\mu_1(x_1), \ldots, \mu_t(x_t)]\}$$

Then applying one of the above methods, they convert the MF_s into its representative value.

Where:
$MF_s \rightarrow$ Mean of fuzzy sets
$x_i \rightarrow$ A set of the standardized base variable
$k_i \rightarrow$ The number of items assigned the linguistic value L_i
$n \rightarrow$ Number of observations in a sample S
$\mu_i(x_i) \rightarrow$ The membership function of F_i

4.2.2. After the conversion

They converted the individual linguistic values of a sample (using one of the above 4 methods), into their representative values and then they calculated the sample mean:

$$M = \frac{1}{n}(r_1k_1 + \cdots + r_tk_t)$$

After that, they calculated the center line: $CL = \frac{1}{m}\sum_{j=1}^{m} M_j$

Where:
$M_j \rightarrow$ Sample mean of the j^{th} sample
$r_i \rightarrow$ The representative value of fuzzy set F_i
Assume that there are $m - samples$ of $size - n$

Based on the interpretation of control charts, Wang and Raz (1990) presented the following approaches of the control limits:

4.2.3. Probabilistic Control Limits

The points plotted on the chart are sample means of representative values in the interval [0,1] with mean of standard deviation $MSD = \frac{1}{m}\sum_{j=1}^{m} SD_j$:

$$\begin{cases} LCL = \max\{0, (CL - A_3 MSD)\} \\ UCL = \min\{1, (CL + A_3 MSD)\} \end{cases}$$

Where:

$SD_j \rightarrow$ The standard deviation for representative values in sample j

4.2.4. Membership Control Limits

The control limits are based on the membership function and the points plotted on the chart are sample means of representative values in the interval [0,1] with mean deviation $\delta = \delta_l + \delta_r$:

$$\begin{cases} LCL = \max\{0, (CL - k\delta)\} \\ UCL = \min\{1, (CL + k\delta)\} \end{cases}$$

Where:

$\delta_l \rightarrow$ The left mean deviation

$\delta_r \rightarrow$ The right mean deviation

4.2.5. Similarities and differences between membership and probabilistic approach

Also, Raz and Wang (1990), presented analytically the construction of control charts for quality data that is available in linguistic form and point out similarities and differences between the two approaches (probabilistic and membership). Both control charts are based on fuzzy set theory and used fuzzy sets to model the linguistic terms. On the other hand the two approaches differ in the extent to which the fuzziness of the data is retained as well as on the way that control limits are defined.

The Table shows the similarities and differences between probabilistic and membership approach:

	Probabilistic	**Membership**
Construction of control charts	1. Using one of the transformation methods to convert the fuzzy sets into its representative values 2. Calculate the sample mean M_j 3. Calculate the st. deviation SD_j 4. Center line: $CL = \frac{1}{m}\sum_{j=1}^{m} M_j$ 5. Mean sample st. deviation MSD 6. Control Limits: $\begin{cases} LCL = \max\{0, (CL - A_3 MSD)\} \\ UCL = \min\{1, (CL + A_3 MSD)\} \end{cases}$	1. Calculate the MF_j for each sample 2. Calculate the grand mean of the sample means: GMF 3. Using one of the transformation methods to convert the GMF into its representative value 4. Center line: representative value of GMF 5. Calculate $\delta(GMF)$ 6. Define the value of k 7. Control Limits $\begin{cases} LCL = \max\{0, (CL - k\delta)\} \\ UCL = \min\{1, (CL + k\delta)\} \end{cases}$
Control Limits	Multiple of the standard deviation which measures the dispersion of the distribution	Multiple of a quality that measures the dispersion of fuzzy sets: insert the notion of mean deviation $\delta = \delta_l + \delta_r$
Fuzziness of the data retained	• Representative values obtained directly from the linguistic terms using one of the transformation methods	• Linguistic terms are not converted directly into representative values (one more step) • Using one of the transformation methods converts the average of the individual linguistic terms into its representative value

Table 4.1. Similarities and differences between membership and probabilistic approach

Where,

$GMF = \frac{1}{m}\sum_{j=1}^{m} MF_j \rightarrow$ The grand mean of the m initially available samples

The purpose of control charts is to detect a shift in the process as soon as it possible. The detection capability of control charts can presented as the complement of $type$ II error. [$type$ II error: The probability, that reject H_1 (assume that there is not any shift in the process level), given that the process is out of control]. The factors whose seem to affect the performance of either control charts were directly connected with $type$ II error. The number of terms in the term set seems to affect the value of $type$ II error: the greater the number of linguistic terms used to classify the observations, the probability of $type$ II error diminishes, therefore there is greater detection capability and sensitivity of control charts.

4.2.6. Comparison between P – chart and proposed control charts

Furthermore, Raz and Wang (1990) compared the performance of two approached control charts for linguistic data with the conventional P – chart and the result is that the performance of the two approached control charts is superior to that of the P – chart. There is more sensitivity in detecting a process shift. This is a result of the fact that, as more information about the process is obtained, the ability of the control chart to detect a process shifts increases.

4.3. Kanagawa et al. Approach

Until this moment, we have seen that, Wang and Raz constructed control charts using linguistic data in order only to control the process average. Kanagawa et al. (1993), developed control charts for linguistic variables based on the estimation of probability distribution existing behind the linguistic data in order to control the process variability as well as the process average.

Firstly, they assumed that they have the following probability density function (pdf) that can be represented by the Gram – Charlier series:

$$f(x) = \varphi(\chi)[1 + a_1 H_1(x) + a_2 H_2(x) + \cdots]$$

Where:

$\varphi(\chi) \rightarrow$ Denotes the pdf of $N(0,1)$

$H_r(x) \rightarrow$ Hermite polynomial of degree r

$b_r \rightarrow$ Moment of degree r

$k_r \rightarrow$ Cumulant of degree r

The probability of a linguistic variable L_i (using Zadeh's):

$$P(L_i) = \int \mu_i(x) f(x) dx$$

In generally, in order to construct the control charts it is necessary to know the pdf of each linguistic variable. There are two cases:

I. *If the membership function* $\mu_i(x)$ *and the pdf* $f(x)$ *are known*

II. *If the membership function* $\mu_i(x)$ *is known and the pdf* $f(x)$ *are unknown*

$\Longrightarrow$ In that case, the estimation of $f(x)$ comes from k_i and μ_i using an iterative algorithm

The construction of control charts by Wang and Raz is based on the conversion method selected. However, Kanagawa et al. (1993) used the conversion method from Zadeh's probability definition (because the data does not have normal distribution): $P(L_i) = \int \mu_i(x) f(x) dx$, and the representative values for linguistic variables are given: $x_i = Rep(F_i) = \frac{\int_{-\infty}^{+\infty} x\mu_i(x)f(x)dx}{\int_{-\infty}^{+\infty} \mu_i(x)f(x)dx}$

4.3.1. Controlling the process average

The control chart for controlling the process average by $1 - \alpha$ confidence limits:

$$\begin{cases} UCL = CL^* + \frac{1}{\sqrt{n}} SD^* \left\{ u_a + \frac{SKW^*}{6\sqrt{n}} \left(u_{a/2}{}^2 - 1 \right) + \frac{KUT^*}{24n} \left(u_{a/2}{}^3 - 3u_{a/2} \right) - \frac{SKW^{*2}}{36n} \left(2u_{a/2}{}^3 - 5u_{a/2} \right) \right\} \\ CL^* = \frac{1}{m} \sum_{j=1}^{m} k_{j1} \\ LCL = CL^* + \frac{1}{\sqrt{n}} SD^* \left\{ u_{1-a/2} + \frac{SKW^*}{6\sqrt{n}} \left(u_{1-a/2}{}^2 - 1 \right) + \frac{KUT^*}{24n} \left(u_{1-a/2}{}^3 - 3u_{1-a/2} \right) - \frac{SKW^{*2}}{36n} \left(2u_{1-a/2}{}^3 - 5u_{1-a/2} \right) \right\} \end{cases}$$

Where:

CL^* → The average mean

SD^{*2} → The average standard deviation

u_a → Upper α – quantile of $N(0.1)$

SKW^* → The average skewness

KUT^* → The average kurtosis

4.3.2. Controlling the process variability

Also, the control chart for controlling the process variability by 1 – α confidence limits:

$$\begin{cases} UCL = \frac{E[SD] y_{a/2}}{v} \\ CL = E[SD] \\ LCL = \frac{E[SD] y_{1-a/2}}{v} \end{cases}$$

Where:

$v = \frac{2[E[SD]]^2}{Var[SD]}$

SD → The average standard deviation

y_a → Upper α – quantile of $g(y)$

$g(y)$ → The pdf of the random variable Y

The advantage of those control charts is that directly controlling the underlying probability distribution $f(x)$ of linguistic data and this makes them more precisely.

4.4. Taleb &Limam Approach

4.4.1. Usages and characteristics of different control charts approaches

Taleb and Limam (2002) presented the following table of usages and characteristics of different control charts approaches (fuzzy and probability):

CONTROL CHART TYPE	Generalized p – chart	Fuzzy control Chart: Probabilistic approach	Fuzzy control Chart: Membership approach	Non – normal fuzzy chart
BASED ON	Probability theory	Fuzzy theory and Probability theory	Fuzzy theory	Fuzzy theory
PURPOSE	A generalization of P – chart to the multinomial process	Use the fuzzy set theory to obtain a real value for each sample. The distribution of these values is supposedly normal	Use the fuzzy set theory to combine all observations in only one fuzzy subset	Using the probability density function behind linguistic data and fuzzy theory to determine the representative values of linguistic terms
CENTRE LINE	No centre line in this approach	It corresponds to the arithmetic mean of representative values of the samples initially available	Corresponds to the representative value of the aggregate fuzzy subset	Corresponds to the average mean of the sample cumulants

CONTROL LIMITS	The upper control limit corresponds to a level of the chi – square distribution	UCL and LCL are determined from the formulae for control chart for variables	UCL and LCL are determined by simulation (k) and using rules of fuzzy arithmetic	Control limits are determined using Gram – Charlier series and probability limit method
SAMPLE SIZE	We will adequate to justify the use of the chi – square distribution	Control limits depend on sample size	Control limits do not depend on sample size	Control limits depend on sample size
REMARKS	Cannot specify if the change in the quality is a result of quality improvement or not	Assignable causes cannot be defined clearly	The computed method of value *k* is not clear. Assignable causes cannot be defined clearly	Unknown probability distribution function cannot be determined easily
BASIC REFERENCES	**Marcucci**	**Raz and Wang**	**Raz and Wang**	**Kanagawa et al.**

Table 4.2. Usages and characteristics of different control charts approaches

4.4.2. Marcucci Approach

Firstly, Taleb and Limam (2002) described the Marcucci approach. Marcucci proposed two approaches using Shewhart type control charts in order to monitor multinomial process when products are classified into mutually exclusive linguistic categories.

Let $X_{i1}, X_{i2}, \dots, X_{it}$ the number of observations in categories $1,2,\dots,t$ respectively for the i^{th} monitoring period. Let $i = 0$ be the base period and n_i be the sample size for monitoring period. The two types of Shewhart control charts are:

4.4.2.1. Type I control chart

Marcucci designed this control chart, in order to detect changes in any of the quality proportions. When the quality proportions $\pi_1, \dots, \pi_t$ are designed to be specific values (a – priori), then Pearson goodness – of – fit is a statistical procedure used to monitor the multinomial process:

$$Y_i^{\;2} = \sum_{j=1}^{t} \frac{\left(x_{ij} - n_i\pi_j\right)^2}{n_i\pi_j}$$

Where:

$\pi_j \rightarrow$ Proportion

$x_{ij} \rightarrow$ The number of observations in categories $j = 1,2,\dots t$

$n_i \rightarrow$ The $i^{th} - sample$

When the process is in – control the asymptotic distribution of $Y_i^{\;2}$ is $\mathcal{X}^2_{(t-1)}$

4.4.2.2. Type II control chart

Marcucci designed this control chart using the multinomial distribution, which can be approximated by a multivariate normal distribution. When the process proportions $\pi_1, \dots, \pi_t$ are estimated for a base period where the process is assumed in – control. When $\pi_1, \dots, \pi_t$ are not specified, then the Pearson goodness – of – fit is not applicable. So, the test of homogeneity of proportions between the base period $(i = 0)$ and each monitoring period i is an appropriate statistical procedure:

$$Z_i^{\;2} = n_i n_0 \sum_{j=1}^{t} \frac{\left(p_{ij} - p_{0j}\right)^2}{x_{ij} + x_{0j}}$$

Where:

$p_{ij} = \frac{x_{kj}}{n_k} \rightarrow$ Sample proportions

$n_i \rightarrow$ Sample size

$k = \{0, i\}$

Then, depending on the value derived, decide whether each sample is in – control or out – of – control.

4.4.3. Comparison between Marcucci and Wang & Raz approaches

Taleb and Limam (2002) wanted to compare those control charts in order to investigate which approach performs better. That succeeded, comparing Marcucci and Raz & Wang approaches using criteria such as sensitivity (samples under – control) and ARL (average run length).

They concluded that fuzzy control charts lead to better results than the generalized P – chart (Marcucci approach) if the membership functions and the transformation method are precisely selected. Also using the probabilistic approach, if the degree of fuzziness of the fuzzy subsets, associated with linguistic terms, is increased, the control chart then becomes more sensitive. Contrary to the conclusions of Raz and Wang (1990), Taleb and Limam (2002), showed that fuzzy control charts affected by the degree of fuzziness and the transformation method used to obtain the representative values.

4.5. Gulbay et al. Approach

Gulbay et al. (2004) proposed an approach differs from previous studies from the point of view of inspection tightness. They constructed α – cut fuzzy control charts for linguistic data to provide the ability of detecting the tightness of the inspection by selecting a suitable α – level.

As in crisp case there are four types of attribute control charts, so here there are the following types of control charts based on:

- ✓ The fraction rejected as non – conforming to specifications
- ✓ The number of non – conforming items
- ✓ The number of non – conformities
- ✓ The number of non – conformities per unit

4.5.1. α – level fuzzy control charts for attributes

Following the steps below the α – cut fuzzy control charts for attributes is constructed:

1. Because CL is a fuzzy set, it can be represented by triangular fuzzy numbers whose fuzzy mode is CL

2. Calculate $L_j(a)$ and $R_j(a)$ for each sample (for each control chart)

3. Determine the membership function of CL

4. Because the membership function of CL is divided into two components,

$$Control\ Limits\ (a) = \begin{cases} Control\ Limits\ (L) \\ Control\ Limits\ (R) \end{cases}$$

5. $Process\ Control = \begin{cases} 1, & (in - control) \\ 0, & (out - of - control) \end{cases}$. This applies to both following situations.

   $ASS \rightarrow$ The average sample size and
   $VSS \rightarrow$ The variable sample size

They concluded that the greater the tightness of the inspection needs for products, the greater must be the value for α – cut. Specifically, they used the phrase *"the higher α, the tighter inspection"*. Also, they concluded that their approach is flexible, not complex, easy in computation, similar to the crisp control charts for attributes and has the ability of detecting out – of – control points at least as effectively as the other approaches do.

4.6. Conclusion

In conclusion, we have seen that using intermediate levels (linguistic terms) to describe the quality product, we have more information about the process, so the ability of the control chart to detect a process shift increases.

Also, in order to retain the standard format of control charts it is necessary to convert the fuzzy sets associated with linguistic data into its representative values. Besides that, the representation of linguistic variables as fuzzy sets retains the ambiguity and vagueness inherent in natural languages and improves the expressive ability of quality assurance inspectors.

In addition, Wand and Raz (1990) concluded, that the factor which affect the performance of control charts is the number of linguistic terms used to classify the observations. Contrary to the above conclusion, Taleb and Limam (2002), showed that fuzzy control charts affected by the degree of fuzziness and the transformation method used to obtain the representative values.

Furthermore, Gulbay et al. (2004) said that the higher α, the tighter inspection (the greater the tightness of the inspection needs for products, the greater must be the value for α – cut).

Finally, Gulbay et al. (2004) presented the following table with a comparison of traditional Shewhart control charts and fuzzy inference control charts:

But, how the membership function of linguistic variables should be constructed, or which should be the appropriate degree of fuzziness, or how many linguistic terms should be defined, are some problems which are still remain unsolved and may will be subjects of future study.

4.6.1. Comparison between Shewhart charts and fuzzy inference charts

	Traditional Shewhart control charts	**Fuzzy inference control charts**
Advantages	1. Easier for considering one quality characteristic 2. More objective	1. Provide more accurate control standards for the process based on expert's experience 2. More flexible for the definitions of the fuzzy inference rules in control charts
Disadvantages	1. Control limits are not flexible 2. Sample size influences the width of control charts 3. Historical data need to be verified to obtain the normal control limits	1. Inference outcomes are based on the subjective experience rules 2. Supplemental rules of the traditional control charts cannot be used

Table 4.3. Comparison between Shewhart charts and fuzzy inference

CHAPTER 5

VARIOUS APPROACHES FOR THE CONSTRUCTION OF FUZZY CONTROL CHARTS

5.1. Introduction

Precise data are not always available. So, the variability cannot be measured with certainty. The theory of fuzzy sets, which first introduced by Zadeh, can adequately model processes where observed data are vague, uncertain, come from human subjectivity, or is available in the form of incomplete information.

Also, the traditional Shewhart control charts interpret information that comes from historical data, completely required and certain. In contrary with traditional Shewhart charts, fuzzy control charts interpret information that comes from experts' experience rules. Fuzzy control charts are inevitable to use when the statistical data in consideration are uncertain or vague or available information about the process is incomplete or includes human subjectivity.

Therefore, many researchers have designed several methods in order to construct fuzzy control charts that accommodate uncertainty due to fuzziness.

In this chapter, we are going to investigate various approaches for the construction of fuzzy control charts that have been suggested by different researchers which are either using defuzzification methods or not. Some of these methods, followed out in detail.

Part I: Approaches without using defuzzification methods

5.2. Cheng's Approach – Construction of fuzzy numbers

As we have seen analytically in the previous chapter, Wang and Raz adopted linguistic terms to express the intermediate levels of a quality characteristic, in order to rate the quality of an inspected item. This assumption seems to be inappropriate when the membership function of the linguistic term is not well constructed.

Due to a limited perception of the values in the rating scale, different inspectors may assign different scores to the same product.

In order to deal with the experts' subjective judgments Cheng (2005), employed fuzzy numbers to aggregate the experts' rating scores to represent the dispersion of the vague observations.

The fuzzy process control methodology proposed by Cheng (2005), comprises an off – line stage and an on – line stage.

5.2.1. Off – Line Stage

Each expert assigns quality ratings to products based on a numerical scale $(0, \ldots, G)$. The individual numerical ratings are then aggregated to form collective opinions expressed in the form of fuzzy numbers.

Having the scores $g_1, \ldots, g_n$ which assigned by $n-$ different experts when rating the same product, and following the steps below, Cheng (2005), made possible the construction of fuzzy numbers (a, m, b).

1. Firstly, calculate the relative distance matrix $D = \left[d_{ij}\right]_{n \times n}$, which presents the distances between of each g_i, where $d_{ij} = \left|g_i - g_j\right|$ and $d_{ii} = 0$, $d_{ij} = d_{ji}$.

2. Then, calculate the average of the relative distances for each g_i:

$$\overline{d_i} = \sum_{j=1}^{n} d_{ij}.$$

3. After that, calculate the pair – wise comparison matrix $\mathrm{P} = \left[p_{ij}\right]_{n \times n}$, which determines the degree of importance of each g_i, where $p_{ij} = \frac{d_j}{d_i}$ is the relative importance of g_i compared to g_j, and $p_{ii} = 1,\ p_{ij} = \frac{1}{p_{ji}}$.
4. Also, calculate the weights associated with each of the scores g_i, such that $w_j = \frac{1}{\sum_{i=1}^{n} p_{ij}}$, $j = 1, \dots n$, and then estimate the mode of the fuzzy number $m = \sum_{i=1}^{n} w_i g_i$.
5. Calculate the g^l and g^r, which are the weighted average of the scores which are less than m and greater than m respectively, and thereafter calculate the ratio of the left spread to the right spread: $\hat{p} = \frac{m - g^l}{g^r - m}$.
6. Finally calculate the left and right endpoints of the fuzzy number a and b respectively.

5.2.2. On – Line Stage

Having the product dimensions come from the off – line stage, and using fuzzy regression analysis it is easy to define automatically the appropriate fuzzy quality ratings.

5.2.2.1. Fuzzy Regression Analysis Model (FRBFN)

In order to retain the aggregated collective knowledge of the experts, Cheng (2005), used the fuzzy regression model – FRBFN (Fuzzy Radial Basis Function Network), proposed by Cheng & Lee (2001), to identify the relationship between the dimensions of a product, $Y = (a, m, b)$, and its fuzzy quality rating, $\hat{Y} = (\hat{a}, \widehat{m}, \hat{b})$.

Assuming that a sample of M fuzzy quality ratings $\widehat{Y_1}, \dots, \widehat{Y_M}$ has been obtained and the estimation of the process mean when the process is thought to be in – control defined as: $\overline{Y} = \frac{1}{M} \sum_{k=1}^{M} \widehat{Y_k}$. Then, plotting the fuzzy quality ratings on the control chart and in order to decide either the process is in – control or out – of – control, make a comparison between $\overline{Y} = \left(\overline{a}, \overline{m}, \overline{b}\right)$ and $\tilde{Y} = \left(\tilde{a}, \tilde{m}, \tilde{b}\right)$ using possibility and necessity measures. In general once can assume that “If the degree of matching between $\tilde{Y}$ and $\overline{Y}$ is high, then the process is considered to be in – control”. Specifically, this is achieved using the following conditions:

5.2.3. Out – of – control conditions

5.2.3.1. Possibility measure

If the possibility measure of the fuzzy sample mean $\tilde{Y}$ is greater than or equal to a threshold parameter α **then** the fuzzy process is considered to be in – control:

$$\{\, if\ Pos(\overline{Y}/\tilde{Y}) \geq a, then\ [\overline{Y}]_{\alpha} \cap \left[\tilde{Y}\right]_{\alpha} \neq \emptyset \text{ ,when } 0 < \alpha \leq 1\}$$

Where,

$[\overline{Y}]_{\alpha}$ → The α – level set

$\tilde{Y} = (\tilde{a}, \tilde{m}, \tilde{b})$ → Mean from a sample drawn from the process

$\overline{Y} = \frac{1}{M}\sum_{k=1}^{M} \hat{Y}_k$ → Mean of the in – control process

$\hat{Y}_k$ → Quality rating score of k – product

Possibility Measure of the variable $\tilde{Y}$ satisfying the condition "$\tilde{Y}\ is\ \overline{Y}$" :

$$Pos(\overline{Y}/\tilde{Y}) = sup_{z \in U}[\min\{\mu_{\overline{Y}}(z), \Pi_{\tilde{Y}}(z)\}]$$

Where,

$\Pi_{\tilde{Y}}$ → Possibility distribution of $\tilde{Y}$

U → Universe

z → Elements of U

5.2.3.2. Necessity measure

If the necessity measure of the fuzzy sample mean $\tilde{Y}$ is greater than or equal to a threshold parameter β **then** the fuzzy process is considered to be in – control:

$$\{ if\ Nec(\bar{Y}/\tilde{Y}) \geq b\ , then\ [\bar{Y}]_b \supset [\tilde{Y}]_{1-b}\ , \text{when } 0 < b \leq 1\}$$

Where,

Necessity Measure of the variable $\tilde{Y}$ satisfying the condition "$\tilde{Y}\ is\ \bar{Y}$" :

$$Nec(\bar{Y}/\tilde{Y}) = inf_{z \in U}[\max\{\mu_{\bar{Y}}(z), 1 - \Pi_{\tilde{Y}}(z)\}]$$

If for a sample mean $\tilde{Y}$, either $[\bar{Y}]_\alpha \cap [\tilde{Y}]_\alpha = \emptyset\ or\ [\bar{Y}]_b \not\supset [\tilde{Y}]_{1-b}$, **then** the fuzzy process is considered to be out – of – control

Cheng (2005), concluded that the possibility measure can indicate the compatibility of the mode of the sample mean with that of the in – control process mean. In the other hand, the necessity measure, not only assesses the necessity of the sample mean $\tilde{Y}$ to conform to the in – control process mean, but also serves as a measure to indicate whether the fuzziness of a sample mean is too large compared to that of the in – control process mean. The thresholds of the possibility and necessity measures, α and β respectively, be established based on a pre – specified probability of $TYPE\ I\ error$, and play important roles in determining the out – of – control conditions, which justify the conformance or otherwise of samples to the process.

Also, the proposed fuzzy process control methodology (FBRFN), in which fuzzy control charts are employed to monitor a process whose outcomes are represented by fuzzy numbers, monitors the central tendency as well as the fuzziness of the process to be monitored.

5.3. Wang Approach

As we have seen above, Cheng (2005), generated a fuzzy number based on a group of experts' scores on a quality item in a quality control process. Here, Wang (2006), assigned a fuzzy number for each outcome of a fuzzy observation on quality monitoring process, assuming that the quality data collected from the fuzzy observation process can be assigned LR – fuzzy quality numbers. The LR – fuzzy quality numbers are used because it is able, easier than fuzzy numbers, to represent simultaneously not only the randomness but also the fuzziness of the fuzzy quality data.

5.3.1. LR – fuzzy quality data

The LR – fuzzy numbers have the form: $X = (m, l, r)_{LR}$

Where,
$m, l, r \rightarrow$ Independent real numbers
$m \rightarrow$ The central point of fuzzy number X
$l > 0 \rightarrow$ The left spread of fuzzy number X
$r > 0 \rightarrow$ The right spread of fuzzy number X

As we have studied in detailed in Chapter 4, Wang & Raz (1990) and Kanagawa et al. (1993), proposed several transformation methods to represent the fuzzy data, by using the fuzzy mode; the α – level fuzzy midrange; the fuzzy median; the fuzzy average and the barycentre concerned with Zadeh's probability measure, in order to retain the standard format of control charts. When comparing the above proposed methods, the conclusions reached are that the methods by using the fuzzy mode and the α – level fuzzy midrange are easier to calculate than the others, however they only took account of the randomness of the fuzzy sample. Also, the method by using the fuzzy median used a non – standard measure of fuzziness thus may be a biased representative of a fuzzy sample. And finally, the method by using the fuzzy average and the barycentre concerned with Zadeh's probability measure are not easy to calculate the representative values.

5.3.2. Optimal Representative Values

It should be noted that the representative values should properly represent randomness and fuzziness simultaneously. Wang (2006), proposed an optimal representative value for LR – fuzzy quality sample (data) by means of a combination of a random variable with a measure of fuzziness, presented as follows:

$$Rep(X) = m + D(X)$$

Analytically,

$X = (m, l, r)_{LR} \rightarrow$ The LR – fuzzy quality sample

$m \rightarrow$ The central variable, which represents the randomness of the LR – fuzzy quality sample

$D(X) = l\beta_1 + r\beta_2 \rightarrow$ The measure of fuzziness, which represents the fuzziness level of the LR – fuzzy quality sample

$m, l, r \rightarrow$ Independent real valued random variables

Also, the representative value for LR – fuzzy quality sample proposed by Wang (2006), can be written as: $Rep(X) = m + l\beta_1 + r\beta_2$

The measure of fuzziness, $D(X)$, for the LR – fuzzy quality sample, $X = (m, l, r)_{LR}$, is an extension of Hamming's measure of fuzziness and presented as:

$$D(X) = \int_{-\infty}^{+\infty} |X(x) - X_{0.5}(x)| dx$$

Where,

$X_{0.5} \rightarrow$ The 0.5 – level set of the fuzzy quality sample X

$X_{0.5}(x) = I_{X_{0.5}}(x) = \begin{cases} 1, x \in X_{0.5} \\ 0, x \notin X_{0.5} \end{cases} \rightarrow$ The indicator of the non – fuzzy quality sample $X_{0.5}$

Because each fuzzy data is characterized by the both randomness and fuzziness, the proposed representative value is considerably accurate, simply, with lower complexity in computation, with an optimal representativeness and very comprehensive because it fully represent the randomness as well as fuzziness measured by a standard fuzziness measure into account.

5.3.3. Construction of a CUSUM chart for LR – fuzzy quality data

Afterwards, Wang (2006), applying the classical CUSUM chart for these representative values, an appropriate and accurate representative CUSUM chart for LR – fuzzy quality data is constructed, using the following steps:

<u>Step 1:</u> Choose a suitable reference value T, here $T = \hat{\mu}$
<u>Step 2:</u> Use the standard scheme: $h = 5\ and\ f = 0.5$
<u>Step 3:</u> Calculate the CUSUM S_n with reference value $K_1 = T + f \cdot \hat{\sigma}_e \geq 0$. Also calculate the CUSUM T_n with reference value $K_2 = T - f \cdot \hat{\sigma}_e < 0$
<u>Step 4:</u> Action is signaled if some $S_n \geq \mathrm{h}\hat{\sigma}_e$ or $T_n \leq -\mathrm{h}\hat{\sigma}_e$

Where,
$\hat{\mu} \rightarrow$ The overall mean of the past observations
$\hat{\sigma}_e = \left(\frac{\sum_{i=1}^{k}(n_i-1)s_i^2}{\sum_{i=1}^{k}(n_i-1)}\right)^{1/2} \rightarrow$ Estimated standard error of samples mean, where, n_i is the group size (there are k – groups each with varying group number), and s_i is the standard error of mean for the representative values in group i

Finally, the fuzzy control charts derived from using the proposed representative method can be improved to some sense.

5.4. Fazel Zarandi et al. Approach

5.4.1. Vague Process Parameters

When we are faced with uncertainty either in process parameters or in the sample data resulting from the unknown nature of the process or from different degree of belief corresponding to various experts, it does not able to assign an exact value to the data. In order to deal with the above vagueness, it is necessary to use fuzzy numbers. Fazel Zarandi et al. (2006) handled the above situation, transforming the vague amount of the process parameters into its equivalent a – cuts and proposing fuzzy control charts for variable and attribute quality characteristics.

5.4.1.1. Variable Control Charts

In the case of variable quality characteristics, Fazel Zarandi et al. (2006), proposed the construction of fuzzy $\overline{X}$ control chart, when the process parameters are fuzzy $(\tilde{m}, \tilde{\sigma})$. Where, $\tilde{m}$ is the fuzzy mean, and $\tilde{\sigma}$ is the fuzzy standard deviation. In order to deal with vagueness in process parameters, it is necessary to use fuzzy numbers, such as L – type fuzzy numbers or triangular fuzzy numbers.

As we have known, in the traditional form of $\overline{X}$ control chart, each point belongs to the control interval, has the form as: $\overline{x} = m + k\sigma$ with $-A \leq k \leq A$. In the case where there is vagueness in the process parameters, each point belongs to the fuzzy interval with more than one degree of membership, and especially Fazel Zarandi et al. (2006) have shown than each point pertains to the interval with its maximum degree of membership.

Because the possibility distribution of fuzzy $\overline{X}$ control chart is a symmetric trapezoidal fuzzy distribution, they transformed the vague amount of the process parameters into its a – cuts of a trapezoidal fuzzy number, and consequently they proposed the following parametric control interval:

$$\begin{cases} UCL(a) = (d_m + Ad_\sigma)(1-\alpha) + c_m + Ac_\sigma \\ LCL(a) = (d_m + Ad_\sigma)(1-\alpha) + c_m - Ac_\sigma \end{cases}$$

Where,

$m, \sigma \rightarrow$ Are triangular fuzzy numbers

$c_m \rightarrow$ The center of fuzzy number m

$c_\sigma \rightarrow$ The center of fuzzy number σ

$d_m \rightarrow$ The width or spread around the center c_m

$d_\sigma \rightarrow$ The width or spread around the center c_σ

5.4.1.2. Attribute Control Charts

In the case of attribute quality characteristics, Fazel Zarandi et al. (2006), proposed the construction of fuzzy U – control chart, when the mean number of defects λ is vague.

Each point in fuzzy U – control chart can be represented as $\tilde{x} = \tilde{\lambda} + 3k\sqrt{\tilde{\lambda}/n}$ with $-1 \leq k \leq 1$. Where, n is the sample size, and $\tilde{\lambda}$ is the fuzzy mean number of defects. For each point in the fuzzy control interval, there is a set of membership functions and also, as in the case of variable quality characteristics, each point belongs to the interval with its maximum degree of membership.

Also, the proposed parametric form of fuzzy U – control interval is:

$$\begin{cases} UCL(a) = c_\lambda + d_\lambda(1-a) + 3\sqrt{\frac{c_\lambda + d_\lambda(1-a)}{n}} \\ LCL(a) = c_\lambda - d_\lambda(1-a) - 3\sqrt{\frac{c_\lambda - d_\lambda(1-a)}{n}} \end{cases}$$

Where,

$\lambda \rightarrow$ Is a triangular fuzzy number, and represents the fuzzy mean number of defects

$c_\lambda \rightarrow$ The center of fuzzy number λ

$d_\lambda \rightarrow$ The width or spread around the center c_λ

$n \rightarrow$ The sample size

Similarly, in the same way resulting the fuzzy C – control chart when $n = 1$.

5.4.2. In the case of linguistic data

As we have seen analytically in previous chapter, in the case where the process data are linguistic, Wang & Raz (1990) in order to calculate the values representing sample mean, they have used a measure of centrality. Contrary with the proposition of Wang & Raz (1990), Fazel Zarandi et al.

(2006) instead of using a measure of centrality they developed a defuzzifier index based on the metric distance between fuzzy sets.

5.4.2.1. Defuzzifier Index

After the introduction of a defuzzifier index, they extended the usage of the above proposed fuzzy control charts, in order to handle the case of linguistic data, following the steps below:

1. Classify each observation with a linguistic value which has a known membership function
2. Obtain a fuzzy set which represents the mean of linguistic terms
3. Use the fuzzy control limits, proposed above by Fazel Zarandi et al. (2006)
4. Use the defuzzifier index which gives the representative value corresponding to the sample mean fuzzy set:

$$\frac{D(U\tilde{C}L\,,\tilde{\overline{x}}_i)}{D(L\tilde{C}L\,,U\tilde{C}L\,)} = \frac{|\overline{UCL}(a)-\overline{x}_i|}{\overline{UCL}(a)-\underline{LCL}(a)}$$

Where,

$D(u\,,v) \rightarrow$ The distance between the fuzzy sets u and v

$\overline{u}(a), \underline{u}(a) \rightarrow$ The a – cuts of fuzzy number u

$\tilde{\overline{x}}_i \rightarrow$ The mean fuzzy set related to the $i^{th} - sample$

$\overline{x}_i \rightarrow$ The representative value of $\tilde{\overline{x}}_i$

Also, the defuzzifier index can be used in the case similar to the above, using instead of $U\tilde{C}L$ the $L\tilde{C}L$ with $\tilde{\overline{x}}_i$.

The metric distance $D(u\,,v)$ using by Fazel Zarandi et al. (2006) in order to create the defuzzifier index, is flexible and easy to calculate. Also, as they noted, *"the metric distance is efficiently uses the information encompassed by the possibility distribution of the mean fuzzy set $\tilde{\overline{x}}_i$, whereas other approaches of computing the representative value of a given fuzzy set do not consider this information"*.

Finally, they concluded that the proposed control intervals are more flexible than the similar crisp case because they are a function of degree of expert's presumption.

5.5. Faraz & Moghadam Approach

In order to control the process average of a variable quality characteristic, Faraz & Moghadam (2007), introduced a new fuzzy control chart that has a WL besides UCL. The WL designed for detecting desired shift in the process, when the UCL controls the process in all.

For the construction of the proposed fuzzy control charts, Faraz & Moghadam (2007) follows the following steps:

1. Select the amount of minimum shift (θ), in the process mean that is important and must be detected

2. Select the warning line (WL), using training data generated from normal distribution, in order to detect a shift in the process mean at least equal to θ, so the sample mean is going to be $\theta + \mu$. This is achieved by following the steps below:
   i. Generate m – samples of size n from out – of – control process and then calculate the sample mean (using the sufficient value of $m = 100000$)
   ii. Calculate $d_i = 1 - \mu^*_{good}(\overline{x_i}),\ i = 1,2,\dots,m$
   Where,
   $d_i \rightarrow$ The statistic for i^{th} sample mean
   $\mu^*_{good}(\overline{x_i}) \rightarrow$ The membership value of linguistic term *good* for the i^{th} sample
   iii. Calculate $\mathrm{WL} = \sum_{i=1}^{m} \frac{d_i}{\mathrm{m}}$

3. Create a base rule for WL alarms: The general idea of the base rule is as follows. If some point draw out of WL successively, but still are

below the UCL, then a shift at least equal to θ is detected. Especially, if two (or three) successive points drawn out of WL, and still are below UCL, a shift at least equal to θ ($\geq \theta$) is occurred.

4. Calculate the false alarm rate (a) and average run length (ARL)
   The false alarm rate $a(2)$ (or $a(3)$), is the proportion of plotting two (or three) successive points out of WL or one point out of UCL, when the process is in – control.
   The average run length is the average number of points that must be plotted before a point indicates an out – of – control condition.

5. Choose the proper base rule that produce minimum false alarm rate and adequate average run length

6. After that, plot $d_i = 1 - \mu^*_{Good}(\overline{x_\iota})$ in the chart with upper control limit $UCL = 1 - \mu^*_{Good}\left(\hat{\mu} + 3\hat{\sigma}_{\hat{\mu}}\right)$
   Where,
   $\hat{\sigma}_{\hat{\mu}} \rightarrow$ The estimation of mean process variability and a WL

Then, Faraz & Moghadam (2007), made a comparison between the new fuzzy control chart and Shewhart $\overline{X}$ – control chart, and they concluded that the fuzzy chart has better power for detecting a specified level of shifts in the process. Also, that new fuzzy control chart is a more practical method for controlling the process average.

Furthermore, they concluded that classifying the observations in the rational groups, provides better neural view to inspectors, of the shifts in the process mean. The new fuzzy method offers different strategic options for company to chose and additionally detects the desire shifts more quickly.

As Aparisi (1997) noted, the importance of a process shift depends on the process capability. If a process is very capable, small process shifts hardly influence the amount of non – conforming items. On the other hand, for a process of small capability even a small shift can produce a large amount of non – conforming items. Therefore, the question that should be clarified is which shift sizes are important for control purposes. But this new fuzzy control chart shows the best value of ARL for detecting the specified level. So, Faraz & Moghadam (2007), concluded that this fuzzy chart is more sensible to small shifts without any complexity augmentation to the chart.

5.6. Amirzadeh et al. Approach

When we have to deal with variable quality characteristics, the traditional P – chart takes time to react to shifts in the production process because of its weak response to small shifts variations in the process mean and variance.

As we have seen above, Fazel Zarandi et al. (2006) dealt with variable quality characteristics using fuzzy valued data in order to construct fuzzy control charts. The constructed control limits by them are fuzzy. A few years later, Amirzadeh et al. (2009) suggested the use of real data and treating the quality as a fuzzy set, they constructed precise control limits which are leading to simple decision making.

They proposed the construction of a new P – control chart, ($\tilde{P}$ – control chart), based on the mean degree of non – conformity, instead of using items which can be either conforming or non – conforming.

5.6.1. Fuzzy degree of non – conformity

Firstly, suppose that X is a standard fuzzy quality characteristic of the product item, which is normally distributed. The fuzzy degree of conformity is defined as:

$$\tilde{C}(X) = \begin{cases} 1, & when\ L \leq X \leq U \\ 0, & otherwise \end{cases}$$

Where,

$L, U \rightarrow$ The lower and upper specification limit respectively

It should be noted that the specification limits $SL: [L, U]$, are not depended on the process and may be set by management, the manufacturing engineers, the customer or product developers or by designers.

Consequently, the fuzzy degree of non – conformity, which is a bounded random variable, is defined as:

$$\widetilde{N}(X) = 1 - \tilde{C}(X)$$

5.6.2. New $\widetilde{P}$ – chart

Then presented the control limits for the mean degree of non – conformity using a triangular membership function, proposed by Amirzadeh et al. (2009). ($\tilde{P}$ – control chart):

$$\begin{cases} UCL = E\left[\overline{\widetilde{N}}\right] + k\sqrt{Var[\overline{\widetilde{N}}]} \\ CL = E\left[\overline{\widetilde{N}}\right] \\ LCL = E\left[\overline{\widetilde{N}}\right] - k\sqrt{Var[\overline{\widetilde{N}}]} \end{cases}$$

Where,

$\overline{\widetilde{N}} = \frac{1}{n}\left[\widetilde{N}(X_1) + \cdots + \widetilde{N}(X_n)\right] \rightarrow$ The mean of random samples from $\widetilde{N}(X)$

$E\left[\overline{\widetilde{N}}\right] = \mu_{\widetilde{N}} \rightarrow$ The mean of $\overline{\widetilde{N}}$

$Var\left[\overline{\widetilde{N}}\right] = \sigma^2_{\widetilde{N}/\sqrt{n}} \rightarrow$ The variance of $\overline{\widetilde{N}}$

5.6.3. Comparison between $\widetilde{P}$ – chart and P – chart

Additionally, Amirzadeh et al. (2009), made a comparison between the new control chart, $\tilde{P}$ – control chart, and the traditional P – control chart and they reached to the following conclusions. The $\tilde{P}$ – chart is sensitive to changes in the process mean and variance, provides much more useful information about the process performance than P – chart, allows operating personnel to take corrective action before any defects are actually produced and also indicates

trouble ahead, whereas the P – chart does not react unless the process has already changed and more non – conforming units are being produced.
Analytically, regarding the latter conclusion, it should be noted that, the degree of non – conformity for a characteristic function of a P – chart takes value either one or zero. Using the specification limits one can observe that the shift $\mu = \mu_0 \rightarrow \mu = \mu_1$ is not quickly perceived than of using $\tilde{P}$ – chart, where its fuzzy degree of non – conformity is a triangular membership function takes value in the interval $[0, 1]$.

5.6.3.1. OC and ARL curves

Also Amirzadeh et al. (2009), compared the two types of control charts (new fuzzy and classical), with respect to the ability of detection shifts in process quality using the OC and ARL curves.

Where,
OC → Operating Characteristic Curve: Is a graphical display of the probability of incorrectly accepting the hypothesis of statistical control against shifts in process quality.
ARL → Average Run Length: Is the average number of points that must be plotted before a point indicates an out – of – control condition.

Using an illustrative application, they showed that for shifts in mean and variance in both cases OC and ARL curves, $\tilde{P}$ – chart are consistently below the OC and ARL for P – chart. This means, that the *type II error* of $\tilde{P}$ – charts, is much lower than the *type II error* of P – charts. Consequently, they concluded that the proposed $\tilde{P}$ – chart are more powerful and efficiently than the traditional P – charts.

And finally, using an appropriate membership function, the $\tilde{P}$ – chart constructed by Amirzadeh et al. (2009), is sensitive not only to changes in the mean of the process but also to changes in the variance of the process, and this gives it an advantages over the chart proposed by Faraz & Moghadam (2007), which is sensitive only to changes in the mean of the process.

5.7. Faraz & Shapiro Approach

The most important contribution of control charts, in the case of crisp data, is their ability to give a straightforward answer to the question if the process is in – control. On the other hand, in the case of fuzzy data, in order to retain the standard form of control charts many inspectors used some defuzzification methods transforming the fuzzy data into its representative values, thereby reducing the information of the original fuzzy sets. Also, different transformation methods may result in different conclusions about the process and consequently, there is not a unique answer to the crisp question. Hence, in the case of fuzzy data, fuzzy control charts must answer the question about how much does the process belong to the in – control state, and those fuzzy control charts must satisfy the following properties:

i. Produce a single and accurate answer to the fuzzy question
ii. Avoid any transformation methods
iii. The chart measure and control limits should be based on both fuzzy and random set theory

Faraz & Shapiro (2010), in order to explain existing fuzziness in data while considering the essential variability between observations, proposed a new approach for constructing fuzzy control charts $\left(\tilde{\bar{\bar{X}}} - \widetilde{S^2}\right)$, that can handle both kinds of uncertainty (randomness and incomplete information). That fuzzy control chart, avoids any defuzzification methods (in order not to reduce the information of the original fuzzy sets) and also it is based on a fuzzy in – control region (FIR) as well as on a grade exclusion measure.

5.7.1. Fuzzy in – control region (FIR)

The construction of the fuzzy in – control region (FIR), used by Faraz & Shapiro (2010), with the significance level $(1-a)$, follows the following steps:

1. Assume that the quality characteristic of a process is a trapezoidal LR – fuzzy random variable $\tilde{X}_{LR} = \{a, b, c, d\}_{LR}$, which is normally distributed $[\tilde{X}_{LR} \sim N(\tilde{m}_{LR}, \sigma^2)]$, having $\mu_{\tilde{X}_{LR}}(x)$ as a membership function.

   Where,
   $\tilde{m}_{LR} = \{m_a, m_b, m_c, m_d\}_{LR} \rightarrow$ The unknown fuzzy mean
   $\sigma^2 \rightarrow$ The crisp variance

2. Estimate the process mean
   Using k – fuzzy subgroups each of size n, the estimation of $\tilde{m}_{LR}$ is:
   $$\hat{\tilde{m}}_{LR} = \{\hat{m}_a, \hat{m}_b, \hat{m}_c, \hat{m}_d\} = \left\{\bar{\bar{a}}, \bar{\bar{b}}, \bar{\bar{c}}, \bar{\bar{d}}\right\}$$

3. Calculate the control limits with the significance level $(1-a)$:
   $$\begin{cases} \widetilde{UCL} = \hat{\tilde{m}}_{LR} + z_{a/2}\frac{\sigma}{\sqrt{n}} \\ \widetilde{LCL} = \hat{\tilde{m}}_{LR} - z_{a/2}\frac{\sigma}{\sqrt{n}} \end{cases}$$
   Where,
   $\sigma^2 \rightarrow$ The crisp variance of $\tilde{X}_{LR}$

4. Estimate the variance
   $$\hat{\sigma}^2 = \bar{s}^2$$
   Where,
   ${s_i}^2 \rightarrow$ The unbiased estimator of variance in the i^{th} subgroup
   $\hat{\sigma} = \frac{\sqrt{\bar{s}^2}}{c_4} \rightarrow$ The unbiased estimator of σ

The FIR, for the fuzzy $\overline{X}$ control chart is the interval:

$$\left\{\bar{\bar{a}}, \bar{\bar{b}} + z_{\frac{a}{2}} \frac{\hat{\sigma}}{\sqrt{n}}, \bar{\bar{c}}, \bar{\bar{d}}\right\}_{LR}$$

The FIR, for the fuzzy S^2 control chart is the interval:

$$[LCL, UCL] = \left[\frac{\bar{s}^2}{n-1} \mathcal{X}^2_{1-\frac{a}{2}, n-1}, \frac{\bar{s}^2}{n-1} \mathcal{X}^2_{a/2, n-1}\right]$$

Where,

$\mathcal{X}^2_{a/2,n-1}, \mathcal{X}^2_{1-\frac{a}{2},n-1}$ → The upper and lower $a/2$ percentage of the chi – squared distribution with $n-1$ degrees of freedom

5.7.2. Graded exclusion measure

Having defined the LR – fuzzy random variables, they constructed the fuzzy in – control region. To avoid problems of the form that a sample fuzzy number belongs to the in – control region in different degree, Faraz & Shapiro (2010), defined the measure of graded inclusion.

The graded inclusion measure is a relationship between fuzzy sets F_1 and F_2, which indicates the degree to which F_1 is contained in F_2:

$$Inc(F_1, F_2) = \inf_{x \in X}[\mu_{F_1}(x) \underset{fuzzy}{\Longrightarrow} \mu_{F_2}(x)]$$

Where,

F_1, F_2 → Fuzzy sets in a universe X

$\underset{fuzzy}{\Longrightarrow}$ → The fuzzy implication operator

$\mu_{F_1}(x), \mu_{F_2}(x)$ → The membership functions of fuzzy sets F_1 and F_2 respectively

Furthermore, the graded exclusion measure is a relationship between fuzzy sets F_1 and F_2, which indicates the degree to which F_1 is excluded from F_2:

$$d = 1 - Inc(F_1, F_2) = 1 - \frac{\int min\left(\mu_{F_1}(x), \mu_{F_2}(x)\right)dx}{\int \mu_{F_1}(x)dx}$$

Where,

F_1, F_2 → Are convex, normal and real fuzzy sets in a universe X

$d \rightarrow$ The fuzzy exclusion measure

Also, Faraz & Shapiro (2010), assuming that $F_1 = \overline{\widetilde{X}_i}$ (subgroup fuzzy means) and $F_2 = FIR$, they have concluded the following conditions about the process:

$$Process\ control = \begin{cases} in - control, & when\ d = 0 \\ out - of - control, & when\ d = 1 \\ otherwise, & 0 < d < 1 \end{cases}$$

As we can see, the higher value of d indicates more severe warnings.

The contribution of the proposed methodology of the construction of fuzzy control charts over a standard method is that the proposed chart monitors the processes considering the uncertainties due to both randomness and incomplete information. Furthermore, the usage of the standard format of control charts, which indicates whether the process is either in – control or out – of – control, may trigger out – of – control signals when the process is actually in – control and these false alarms may cause users to lose confidence in control charts. Having the range $0 < d < 1$ using the proposed control chart, there is an extended alarm zone between the in – control to the out – of – control state, which helps to facilitate corrective actions in anticipation of the process subsequently going out – of – control.

Also, Faraz & Shapiro (2010), concluded that the proposed approach is easy to use and calculate. Also, it satisfies the three mentioned properties of fuzzy control charts and takes precedence over the existing fuzzy control charts, having the following main advantages:

1. The chart measure produces a single and precise answer to the fuzzy question ("How much does the process belong to the in – control state") through the degree to which process samples are excluded from the FIR [Satisfies the property i]

2. Both sample fuzzy sets and the in – control region are independent of transformation methods and they are completely compared in a fuzzy space [Satisfies the property ii]
3. A decision about process states is made through fuzzy and random sets, and the proposed approach also has the advantage of simplicity in the field [Satisfies the property iii]

Part II: Approaches using defuzzification methods

5.8. Gulbay & Kahraman Approach

When we have to deal with linguistic data, in order to retain the standard form of control charts and construct fuzzy control limits, it is necessary to transform the fuzzy sets associated with linguistic data into its representative values. This can be achieved with various ways using one of the following transformation methods, similar to the measure of central tendency, based on fuzzy mode; fuzzy midrange; fuzzy average; fuzzy median, etc.

Using the fuzzy transformation methods with a – cut, which provides the ability of determining the tightness of the inspection, β, which is a predefined acceptable percentage, and a trapezoidal fuzzy number (a^a, b, c, d^a), where $a^a = a + a(b-a)$ and $d^a = d - a(d-c)$, Gulbay & Kahraman (2007), developed the following fuzzy control charts. The α – cut is a non – fuzzy set which comprises all elements whose membership degrees are greater or equal to a. Moreover, α – cut interpreted as the tightness of the inspection, *"The higher the value of α the tighter inspection"*. Also, in order to set the tightness of the inspection, it is necessary to define the values of α and β.

5.8.1. Fuzzy Control Charts

5.8.1.1. Based on fuzzy mode transformation

Control Limits: $$\begin{cases} UCL_{mod} = CL_{mod} + 3\sqrt{CL_{mod}} = [UCL_2, UCL_3] \\ CL_{mod} = f_{mod}(\widetilde{CL}) = [CL_2, CL_3] \\ LCL_{mod} = CL_{mod} - 3\sqrt{CL_{mod}} = [LCL_2, LCL_3] \end{cases}$$

The conditions of process control for each sample are presented below:

$$Process\ Control = \begin{cases} in-control, \ \ for\ \beta = 1\ \left[b_j \geq LCL_2 \wedge c_j \leq UCL_3\right] \\ out-of-control, \ \ for\ \beta = 0\ \left[b_j \geq UCL_3 \vee c_j \leq LCL_2\right] \\ otherwise, \ \begin{cases} rather\ in-control, & for\ b_j \geq \beta \\ rather\ out-of-control, & for\ b_j < \beta \end{cases} \end{cases}$$

Where,

$S_{mod,j} = \left[b_j, c_j\right] \rightarrow$ The fuzzy mode of sample j, in the case of trapezoidal fuzzy number with a – cut: (a^a, b, c, d^a) with $a^a = a + a(b-a)$ and $d^a = d - a(d-c)$

$f_{mod} = \{\mathrm{x} \in \mathrm{X} | \mu_f(x) = 1\}$ → Is the fuzzy mode of the fuzzy set f

$\widetilde{CL} = (CL_1^a, CL_2, CL_3, CL_4^a)$ → The center line of the general form of a – level fuzzy control charts

5.8.1.2. Based on a – level fuzzy midrange transformation

Control Limits: $$\begin{cases} UCL_{mr}^a = CL_{mr}^a + 3\sqrt{CL_{mr}^a} \\ CL_{mr}^a = f_{mr}^a(\widetilde{CL}) = \frac{1}{2}(CL_1^a + CL_4^a) \\ LCL_{mr}^a = CL_{mr}^a - 3\sqrt{CL_{mr}^a} \end{cases}$$

The conditions of process control for each sample are presented below:

$$Process\ Control = \begin{cases} in-control, & for\ LCL_{mr}^a \leq S_{mr,j}^a \leq UCL_{mr}^a \\ out-of-control, & otherwise \end{cases}$$

Where,

$S_{mr,j}^a = \frac{1}{2}\left(a_j^a + d_j^a\right)$ → The a – level fuzzy midrange of sample j

$f_{mr}^a = \frac{1}{2}(a^a + d^a)$ → Is the midpoint of the end of the a – cut

5.8.1.3. Based on a – level fuzzy median transformation

Control Limits: $$\begin{cases} UCL_{med}^{a} = CL_{med}^{a} + 3\sqrt{CL_{med}^{a}} \\ CL_{med}^{a} = f_{med}^{a}(\widetilde{CL}) = \frac{1}{4}(CL_1^{a} + CL_2 + CL_3 + CL_4^{a}) \\ LCL_{med}^{a} = CL_{med}^{a} - 3\sqrt{CL_{med}^{a}} \end{cases}$$

The conditions of process control for each sample are presented below:

$$Process\ Control = \begin{cases} in-control, & for\ LCL_{med}^{a} \leq S_{med,j}^{a} \leq UCL_{med}^{a} \\ out-of-control, & otherwise \end{cases}$$

Where,

$S_{med,j}^{a} = \frac{1}{4}\left(a_j^{a} + b_j + c_j + d_j^{a}\right) \rightarrow$ The a – level fuzzy median of sample j

$f_{mr}^{a} \rightarrow$ Is the point which partitions the membership function of a fuzzy set into two equal regions at a – level

When the linguistic data represented by symmetric fuzzy numbers, the various defuzzification methods become equal to each other and therefore, give the same control decisions. On the other hand, when the linguistic data represented by asymmetric fuzzy numbers, different possible decisions can be faced.

5.8.2. Direct Fuzzy Approach

In order to prevent the loss of information included by the fuzzy samples, Gulbay & Kahraman (2007), proposed an alternative approach, DFA – Direct Fuzzy Approach, to construct fuzzy control limits where does not require the use of defuzzification. In their approach, the linguistic data are not transformed into representative values using any transformation method, but compared directly in fuzzy space.

Firstly, they determined the a – level fuzzy control limits by fuzzy arithmetic which have the classical form:

$$\begin{cases} \widetilde{UCL^a} = \widetilde{CL^a} + 3\sqrt{\widetilde{CL^a}} = [UCL_1^a, UCL_2, UCL_3, UCL_4^a] \\ \widetilde{CL^a} = \left(\overline{a^a}, \bar{b}, \bar{c}, \overline{d^a}\right) = [CL_1^a, CL_2, CL_3, CL_4^a] \\ \widetilde{LCL^a} = \widetilde{CL^a} + 3\sqrt{\widetilde{CL^a}} = [LCL_1^a, LCL_2, LCL_3, LCL_4^a] \end{cases}$$

Where,

$\overline{a^a}$ → The arithmetic mean of a^a (similarly and the other)

DFA provides the ability of making linguistic decisions like ''rather in control'' or ''rather out of control''. Further intermediate levels of process control decisions are also possible to introduce:

$$Process\ Control = \begin{cases} \boldsymbol{in - control}, when\ fuzzy\ sample\ completely\ involved\ by\ \widetilde{UCL}\ and\ \widetilde{LCL} \\ \boldsymbol{out - of - control}, when\ fuzzy\ sample\ totally\ excluded\ by\ \widetilde{UCL}\ and\ \widetilde{LCL} \\ otherwise, \begin{cases} \boldsymbol{rather\ in - control}, when\ b_j \geq b \\ \boldsymbol{rather\ out - of - control}, when\ b_j < b \end{cases} \end{cases}$$

Where,

$b_j^a = \frac{S_j^a - A_{out,j}^a}{S_j^a}$ → The percentage sample area within the control limits

S_j^a → The sample's area at a – level

$A_{out} = A_{out}^U + A_{out}^L$ → The total area outside the fuzzy control limits

A_{out}^U → The sample's area above the upper control limits

A_{out}^L → The sample's area falling below the lower control limits

Having presented the three types of fuzzy control charts using different transformation methods, as well and the proposed fuzzy control chart using DFA (based on a fuzzy comparison method), Gulbay & Kahraman (2007), concluded that the proposed approach is very flexible and more accurate than the approaches using fuzzy transformation methods, since both the linguistic data and control limits are not transformed into representative values, in order to prevent the loss of information included in the samples.

5.9. Erginel Approach

When the measurements in a manufacturing process are presented with vague or uncertain observations, fuzzy control charts is an appropriate tool in order to evaluate those vague data. Those uncertainties may come from operators, gauges or environmental conditions.

Erginel (2008), proposed the construction of fuzzy control limits for individual (X) and moving range (MR) control charts with α – cuts by using α – level fuzzy median transformation techniques. The purpose of using α – cuts is to provide the flexibility of control limits and also, the purpose of using the fuzzy median transformation technique instead of others, is that median represents the middle value of a membership function, so is more suitable for individual samples.

Erginel (2008), follows the following steps to construct the fuzzy control limits:

1. Collect the data from a process in the form of triangular fuzzy number (X_a, X_b, X_c)
2. Develop the formulation of fuzzy individual (X) and fuzzy moving range (MR) center line, upper and lower limits
3. Integrate the a – cuts to fuzzy X and fuzzy MR control charts
4. Evaluate the process with a – cuts base on an a – level fuzzy median transformation technique.

The fuzzy control charts as well as the conditions of process control proposed by Erginel (2008), appear below:

5.9.1. Fuzzy Individual Control Chart $(\widetilde{X})$ with α – cuts based on α – level fuzzy median transformation technique

Firstly, because of the use of fuzzy triangular numbers, the fuzzy X control chart is ($\tilde{X}$ – control chart):

$$\begin{cases} \widetilde{UCL} = \widetilde{CL} + 3\frac{\overline{MR}}{d_2} \\ \widetilde{CL} = (\overline{X}_a, \overline{X}_b, \overline{X}_c) = \left(\widetilde{CL}_1, \widetilde{CL}_2, \widetilde{CL}_3\right) \\ \widetilde{LCL} = \widetilde{CL} - 3\frac{\overline{MR}}{d_2} \end{cases}$$

Where,

$\widetilde{CL} = (\overline{X}_a, \overline{X}_b, \overline{X}_c) \rightarrow$ The mean of fuzzy samples

$\overline{MR} = \left(\overline{MR}_a, \overline{MR}_b, \overline{MR}_c\right) \rightarrow$ The mean of fuzzy moving ranges for fuzzy samples

Then, applying a – cuts of fuzzy sets

$X_a^a = X_a + a(X_b - X_a)$ and $X_c^a = X_c - a(X_c - X_b)$,

$\overline{MR}_a^a = \overline{MR}_a + a\left(\overline{MR}_b - \overline{MR}_a\right)$ and $\overline{MR}_c^a = \overline{MR}_c - a\left(\overline{MR}_c - \overline{MR}_b\right)$

the fuzzy X control chart with a – cuts is:

$$\begin{cases} \widetilde{UCL}^a = \widetilde{CL}^a + \frac{3}{d_2}\overline{MR}^a \\ \widetilde{CL}^a = \left(\overline{X}_a^a, \overline{X}_b, \overline{X}_c^a\right) \\ \widetilde{LCL}^a = \widetilde{CL}^a - \frac{3}{d_2}\overline{MR}^a \end{cases}$$

Where,

$\widetilde{CL}^a = \left(\overline{X}_a^a, \overline{X}_b, \overline{X}_c^a\right) \rightarrow$ The mean of fuzzy samples with a – cuts

$\overline{MR}^a = \left(\overline{MR}_a^a, \overline{MR}_b, \overline{MR}_c^a\right) \rightarrow$ The mean of fuzzy moving ranges for fuzzy samples with a – cuts

And finally, the fuzzy X control chart with α – cuts based on α – level fuzzy median transformation technique is:

$$\begin{cases} UCL_{med-X}^{a} = CL_{med-X}^{a} + \frac{3}{d_2}\left[\frac{1}{3}\left(\overline{MR}_a^{a} + \overline{MR}_b + \overline{MR}_c^{a}\right)\right] \\ CL_{med-X}^{a} = f_{med-X}^{a}\left(\widetilde{CL}\right) = \frac{1}{3}\left[CL_{(X)1}^{a} + CL_{(X)2} + CL_{(X)3}^{a}\right] \\ LCL_{med-X}^{a} = CL_{med-X}^{a} - \frac{3}{d_2}\left[\frac{1}{3}\left(\overline{MR}_a^{a} + \overline{MR}_b + \overline{MR}_c^{a}\right)\right] \end{cases}$$

The conditions of process control for each sample:

$$Process\ Control = \begin{cases} in-control, & when: LCL_{med-X}^{a} \leq S_{med-X,j}^{a} \leq UCL_{med-X}^{a} \\ out-of-control, & otherwise \end{cases}$$

Where,

$S_{med-X,j}^{a} = \frac{1}{3}\left[X_{aj}^{a} + X_{bj} + X_{cj}^{a}\right]$ → The α – level fuzzy median for sample j

(X_a, X_b, X_c) → Fuzzy numbers

$(\overline{X}_a, \overline{X}_b, \overline{X}_c)$ → Means of fuzzy numbers for individual data

$\left(\overline{MR}_a, \overline{MR}_b, \overline{MR}_c\right)$ → Means of fuzzy numbers for moving range data

5.9.2. Fuzzy Moving Range Control Chart $\left(\widetilde{MR}\right)$ with α – cuts based on α – level fuzzy median transformation technique

Firstly, because of the use of fuzzy triangular numbers, the fuzzy MR control chart is ($\widetilde{MR}$ – control chart):

$$\begin{cases} \widetilde{UCL} = D_4\overline{MR} \\ \widetilde{CL} = \overline{MR} = \left(\overline{MR}_a, \overline{MR}_b, \overline{MR}_c\right) \\ \widetilde{LCL} = D_3\overline{MR} \end{cases}$$

Where,

$\overline{MR} = \left(\overline{MR}_a, \overline{MR}_b, \overline{MR}_c\right)$

Then, applying a – cuts of fuzzy sets, the fuzzy MR control chart with a – cuts is:

$$\begin{cases} \widetilde{UCL}^{a} = D_4 \overline{MR}^{a} \\ \widetilde{CL}^{a} = \overline{MR}^{a} \\ \widetilde{LCL}^{a} = D_3 \overline{MR}^{a} \end{cases}$$

Where,

$\overline{MR}^{a} = \left(\overline{MR}_a^a, \overline{MR}_b, \overline{MR}_c^a\right)$

And finally, the fuzzy MR control chart with α – cuts based on α – level fuzzy median transformation technique is:

$$\begin{cases} UCL_{med-MR}^{a} = D_4 \left[\frac{1}{3}\left(\overline{MR}_a^a + \overline{MR}_b + \overline{MR}_c^a\right)\right] \\ CL_{med-MR}^{a} = f_{med-MR}^{a}\left(\widetilde{CL}\right) = \frac{1}{3}\left[\overline{MR}_a^a + \overline{MR}_b + \overline{MR}_c^a\right] \\ LCL_{med-MR}^{a} = D_3 \left[\frac{1}{3}\left(\overline{MR}_a^a + \overline{MR}_b + \overline{MR}_c^a\right)\right] \end{cases}$$

The conditions of process control for each sample:

$$Process\ Control = \begin{cases} in-control, & when{:}\ LCL_{med-MR}^{a} \leq S_{med-MR,j}^{a} \leq UCL_{med-MR}^{a} \\ out-of-control, & otherwise \end{cases}$$

Where,

$S_{med-MR,j}^{a} = \frac{1}{3}\left[\overline{MR}_{aj}^a + \overline{MR}_{bj} + \overline{MR}_{cj}^a\right]$ → α – level fuzzy median for sample j

$\left(\overline{MR}_a, \overline{MR}_b, \overline{MR}_c\right)$ → Means of fuzzy numbers for moving range data

Then, Erginel (2008), making a comparison between the proposed fuzzy control limits and the conventional control limits, and concluded that, although there is ambiguity in the process observations, the constructed fuzzy control limits provide more flexibility compared with the conventional control limits.

5.10. Senturk & Erginel Approach

As we have seen above, Erginel (2008), in order to handle the uncertainty which appears on the observations comes from the measurement system including operators and gauges, and environmental conditions, proposed the construction of fuzzy control charts for individual and moving range data. Also, to deal with such situations, Senturk & Erginel (2009), proposed the construction of fuzzy $\bar{X}-R$ and $\bar{X}-S$ control charts for variable quality characteristics with a – cuts by using a – level fuzzy midrange transformation technique.

Senturk & Erginel (2009), follow the following steps for the construction of fuzzy control limits:

1. Using triangular fuzzy numbers, transform the traditional $\bar{X}-R$ and $\bar{X}-S$ control charts to the fuzzy control charts: $\tilde{\bar{X}}-\tilde{R}$ and $\tilde{\bar{X}}-\tilde{S}$
2. Develop $\tilde{\bar{X}}-\tilde{R}$ and $\tilde{\bar{X}}-\tilde{S}$ control charts with a – cuts
3. Calculate $\tilde{\bar{X}}-\tilde{R}$ and $\tilde{\bar{X}}-\tilde{S}$ control charts with a – cuts by using a – level fuzzy midrange transformation technique

The fuzzy control charts as well as the conditions of process control proposed by Erginel (2008), appear below:

5.10.1. Fuzzy $\bar{X}$ control chart based on the ranges R with a – cuts by using a – level fuzzy midrange transformation technique

Firstly, because of the use of fuzzy triangular numbers (X_a, X_b, X_c), the fuzzy $\bar{X}$ control chart is ($\tilde{\bar{\bar{X}}}$ – control chart):

$$\begin{cases} \widetilde{UCL} = \widetilde{CL} + A_2\overline{R} \\ \widetilde{CL} = (\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c) = \left(\widetilde{CL}_1, \widetilde{CL}_2, \widetilde{CL}_3\right) \\ \widetilde{LCL} = \widetilde{CL} - A_2\overline{R} \end{cases}$$

Where,

$\overline{R} = \left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right) \rightarrow$ The average of $R_i's$, where R_i are the ranges of samples

$\overline{R}_a \rightarrow$ The arithmetic mean of the least possible value

$\overline{R}_b \rightarrow$ The arithmetic mean of the most possible value

$\overline{R}_c \rightarrow$ The arithmetic mean of the largest possible value

$\widetilde{CL} = \left(\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c\right) \rightarrow$ The arithmetic mean of fuzzy samples

$A_2 \rightarrow$ Control chart coefficient

Then, applying a – cuts of fuzzy sets

$\overline{\overline{X_a^a}} = \overline{\overline{X_a}} + a\left(\overline{\overline{X_b}} - \overline{\overline{X_a}}\right)$ and $\overline{\overline{X_c^a}} = \overline{\overline{X_c}} - a\left(\overline{\overline{X_c}} - \overline{\overline{X_b}}\right)$,

$\overline{R}_a^a = \overline{R}_a + a\left(\overline{R}_b - \overline{R}_a\right)$ and $\overline{R}_c^a = \overline{R}_c - a\left(\overline{R}_c - \overline{R}_b\right)$

the fuzzy $\overline{X}$ control chart with a – cuts is:

$$\begin{cases} \widetilde{UCL}^a = \widetilde{CL}^a + A_2\overline{R^a} \\ \widetilde{CL}^a = \left(\overline{\overline{X_a^a}}, \overline{\overline{X_b}}, \overline{\overline{X_c^a}}\right) \\ \widetilde{LCL}^a = \widetilde{CL}^a - A_2\overline{R^a} \end{cases}$$

Where,

$\widetilde{CL}^a = \left(\overline{\overline{X_a^a}}, \overline{\overline{X_b}}, \overline{\overline{X_c^a}}\right) \rightarrow$ The mean of fuzzy samples with a – cuts

$\overline{R^a} = \left(\overline{R}_a^a, \overline{R}_b, \overline{R}_c^a\right) \rightarrow$ The mean of fuzzy ranges for fuzzy samples with a – cuts

And finally, the fuzzy $\overline{X}$ control chart based on ranges with α – cuts based on α – level fuzzy median transformation technique is:

$$\begin{cases} UCL_{mr-\overline{X}}^a = CL_{mr-\overline{X}}^a + A_2\left[\frac{1}{2}\left(\overline{R}_a^a + \overline{R}_c^a\right)\right] \\ CL_{mr-\overline{X}}^a = f_{mr-X}^a\left(\widetilde{CL}\right) = \frac{1}{2}\left[CL_{(X)1}^a + CL_{(X)3}^a\right] \\ LCL_{mr-\overline{X}}^a = CL_{mr-\overline{X}}^a - A_2\left[\frac{1}{2}\left(\overline{R}_a^a + \overline{R}_c^a\right)\right] \end{cases}$$

The conditions of process control for each sample:

$$Process\ Control = \begin{cases} in - control, for\ LCL^{a}_{mr-\bar{X}} \leq S^{a}_{mr-\bar{X},j} \leq UCL^{a}_{mr-\bar{X}} \\ out - of - control, otherwise \end{cases}$$

Where,

$S^{a}_{mr-\bar{X},j}$ → The α – level fuzzy midrange of sample j

$f^{a}_{mr} = \frac{1}{2}(a^{a} + c^{a})$ → The midpoint of the ends of the a – level cuts

$a^{a}\ and\ c^{a}$ → The end points of the A^{a}

A^{a} → The α – level cut

5.10.2. Fuzzy $\boldsymbol{R}$ control chart with $\boldsymbol{a}$ – cuts by using $\boldsymbol{a}$ – level fuzzy midrange transformation technique

Firstly, the fuzzy R control chart using fuzzy triangular numbers is:

$$\begin{cases} \widetilde{UCL} = D_4\overline{R} \\ \widetilde{CL} = \overline{R} = \left(\overline{R}_a, \overline{R}_b, \overline{R}_c\right) \\ \widetilde{LCL} = D_3\overline{R} \end{cases}$$

Where,

D_3, D_4 → Are control chart coefficients

Then, applying a – cuts of fuzzy sets

$\overline{R}^{a}_{a} = \overline{R}_a + a\left(\overline{R}_b - \overline{R}_a\right)$ and $\overline{R}^{a}_{c} = \overline{R}_c - a\left(\overline{R}_c - \overline{R}_b\right)$

the fuzzy R control chart with a – cuts is:

$$\begin{cases} \widetilde{UCL}^{a} = D_4\overline{R^{a}} \\ \widetilde{CL}^{a} = \overline{R^{a}} = \left(\overline{R}^{a}_{a}, \overline{R}_b, \overline{R}^{a}_{c}\right) \\ \widetilde{LCL}^{a} = D_3\overline{R^{a}} \end{cases}$$

And finally, the fuzzy R control chart with α – cuts based on α – level fuzzy median transformation technique is:

$$\begin{cases} UCL^{a}_{mr-R} = D_4 f^{a}_{mr-R}(\widetilde{CL}) \\ CL^{a}_{mr-R} = f^{a}_{mr-R}(\widetilde{CL}) = \frac{1}{2}(\overline{R}^{a}_{a} + \overline{R}^{a}_{c}) \\ LCL^{a}_{mr-R} = D_3 f^{a}_{mr-R}(\widetilde{CL}) \end{cases}$$

The conditions of process control for each sample:

$$Process\ Control = \begin{cases} in-control, for\ LCL^{a}_{mr-R} \leq S^{a}_{mr-R,j} \leq UCL^{a}_{mr-R} \\ out-of-control, otherwise \end{cases}$$

Where,

$S^{a}_{mr-R,j} \rightarrow a$ – level fuzzy midrange of sample j for fuzzy $\tilde{R}$ control chart

$D_3\ and\ D_4 \rightarrow$ Control chart coefficients

5.10.3. Fuzzy $\overline{X}$ control chart based on the standard deviation S with a – cuts by using a – level fuzzy midrange transformation technique

Firstly, because of the use of fuzzy triangular numbers (X_a, X_b, X_c), the fuzzy $\overline{X}$ control chart is ($\tilde{\overline{\overline{X}}}$ – control chart):

$$\begin{cases} \widetilde{UCL} = \widetilde{CL} + A_3\overline{S} \\ \widetilde{CL} = (\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c) = (\widetilde{CL}_1, \widetilde{CL}_2, \widetilde{CL}_3) \\ \widetilde{LCL} = \widetilde{CL} - A_3\overline{S} \end{cases}$$

Where,

$\overline{S} = (\overline{S}_a, \overline{S}_b, \overline{S}_c) \rightarrow$ The average of $S_i's$, where S_i are the standard deviation of samples

$\overline{S}_a \rightarrow$ The arithmetic mean of the least possible value

$\overline{S}_b \rightarrow$ The arithmetic mean of the most possible value

$\overline{S}_c \rightarrow$ The arithmetic mean of the largest possible value

$\widetilde{CL} = (\overline{\overline{X}}_a, \overline{\overline{X}}_b, \overline{\overline{X}}_c) \rightarrow$ The arithmetic mean of fuzzy samples

$A_3 \rightarrow$ Control chart coefficient

Then, applying a – cuts of fuzzy sets

$\overline{\overline{X_a^a}} = \overline{\overline{X_a}} + a\left(\overline{\overline{X_b}} - \overline{\overline{X_a}}\right)$ and $\overline{\overline{X_c^a}} = \overline{\overline{X_c}} - a\left(\overline{\overline{X_c}} - \overline{\overline{X_b}}\right)$,

$\overline{S}_a^a = \overline{S}_a + a\left(\overline{S}_b - \overline{S}_a\right)$ and $\overline{S}_c^a = \overline{S}_c - a\left(\overline{S}_c - \overline{S}_b\right)$

the fuzzy $\overline{X}$ control chart with a – cuts is:

$$\begin{cases} \widetilde{UCL}^a = \widetilde{CL}^a + A_3\overline{S^a} \\ \widetilde{CL}^a = \left(\overline{\overline{X_a^a}}, \overline{\overline{X_b}}, \overline{\overline{X_c^a}}\right) \\ \widetilde{LCL}^a = \widetilde{CL}^a - A_3\overline{S^a} \end{cases}$$

Where,

$\widetilde{CL}^a = \left(\overline{\overline{X_a^a}}, \overline{\overline{X_b}}, \overline{\overline{X_c^a}}\right) \rightarrow$ The mean of fuzzy samples with a – cuts

$\overline{S^a} = \left(\overline{R}_a^a, \overline{R}_b, \overline{R}_c^a\right) \rightarrow$ The mean of fuzzy standard deviation for fuzzy samples with a – cuts

And finally, the fuzzy $\overline{X}$ control chart based on the standard deviation with α – cuts based on α – level fuzzy median transformation technique is:

$$\begin{cases} UCL_{mr-\overline{X}}^a = CL_{mr-\overline{X}}^a + A_3\left[\frac{1}{2}\left(\overline{S}_a^a + \overline{S}_c^a\right)\right] \\ CL_{mr-\overline{X}}^a = f_{mr-X}^a\left(\widetilde{CL}\right) = \frac{1}{2}\left[CL_{(X)1}^a + CL_{(X)3}^a\right] \\ LCL_{mr-\overline{X}}^a = CL_{mr-\overline{X}}^a - A_3\left[\frac{1}{2}\left(\overline{S}_a^a + \overline{S}_c^a\right)\right] \end{cases}$$

The conditions of process control for each sample:

$$Process\ Control = \begin{cases} in-control, for\ LCL_{mr-\bar{X}}^a \leq S_{mr-\bar{X},j}^a \leq UCL_{mr-\bar{X}}^a \\ out-of-control, otherwise \end{cases}$$

Where,

$S_{mr-\bar{X},j}^a \rightarrow$ The α – level fuzzy midrange of sample j

5.10.4. Fuzzy S control chart with a – cuts by using a – level fuzzy midrange transformation technique

Firstly, the fuzzy S control chart using fuzzy triangular numbers is:

$$\begin{cases} \widetilde{UCL} = B_4\overline{S} \\ \widetilde{CL} = \overline{S} = \left(\overline{S}_a, \overline{S}_b, \overline{S}_c\right) \\ \widetilde{LCL} = B_3\overline{S} \end{cases}$$

Where,

$B_3, B_4 \rightarrow$ Are control chart coefficients

Then, applying a – cuts of fuzzy sets

$\overline{S}_a^a = \overline{S}_a + a\left(\overline{S}_b - \overline{S}_a\right)$ and $\overline{S}_c^a = \overline{S}_c - a\left(\overline{S}_c - \overline{S}_b\right)$

the fuzzy S control chart with a – cuts is:

$$\begin{cases} \widetilde{UCL}^a = B_4\overline{S^a} \\ \widetilde{CL}^a = \overline{S^a} = \left(\overline{S}_a^a, \overline{S}_b, \overline{S}_c^a\right) \\ \widetilde{LCL}^a = B_3\overline{S^a} \end{cases}$$

And finally, the fuzzy S control chart with α – cuts based on α – level fuzzy median transformation technique is:

$$\begin{cases} UCL_{mr-S}^a = B_4 f_{mr-S}^a\left(\widetilde{CL}\right) \\ CL_{mr-S}^a = f_{mr-S}^a\left(\widetilde{CL}\right) = \frac{1}{2}\left(\overline{S}_a^a + \overline{S}_c^a\right) \\ LCL_{mr-S}^a = B_3 f_{mr-S}^a\left(\widetilde{CL}\right) \end{cases}$$

The conditions of process control for each sample:

$$Process\ Control = \begin{cases} in-control, for\ LCL_{mr-S}^a \leq S_{mr-S,j}^a \leq UCL_{mr-S}^a \\ out-of-control, otherwise \end{cases}$$

Where,

$S^{a}_{mr-S,j} \rightarrow a$ – level fuzzy midrange of sample j for fuzzy $\tilde{S}$ control chart

B_3 and $B_4 \rightarrow$ Control chart coefficients

It should be noted that, when the observations are crisp data, there is a possibility of a sample of the process to be too close to the traditional control limits and may cause false alarm. But, when fuzzy observations are used, Senturk & Erginel (2009), concluded that fuzzy control limits seems to provide more flexibility and accuracy for evaluation and controlling a process.

5.11. Kahraman et al. Approach

As Erginel (2008) and Senturk & Erginel (2009), so and Kahraman et al. (2010), dealt with vague or uncertain observations, and proposed the construction not only for fuzzy variable control charts but also for fuzzy attribute control charts.

As we have seen above in the previous approaches, in order to retain the standard format of control charts, Kahraman et al. (2010) also, converted the fuzzy data into its representative values using defuzzification methods based on a – cuts.

They presented the construction of the following fuzzy attribute control charts using a – level fuzzy median transformation with a – cut. The fuzzy P control chart based on constant (or variable) sample size, the fuzzy nP control chart based on constant sample size, the fuzzy C control chart and finally the fuzzy U control chart.

These control charts were constructed using the same logic followed by the Erginel (2008) and Senturk & Erginel (2009). Also, Kahraman et al. (2010) concluded that using fuzzy numbers to construct control charts provides flexibility for control limits.

5.12. Faraz et al. Approach

When we have to deal with crisp observations, it is easy to monitor and evaluate a process as "in – control" or "out – of – control". For example, it is easy to obtain the value of a sample mean and therefore the value of a crisp statistic $z = \frac{\overline{X} - m_0}{\sigma/\sqrt{n}}$ and also, to use the hypothesis testing because we can clearly distinguish between $H_0 : m = m_0$ and $H_1 : m \neq m_0$.

On the other hand, in real situations there is often uncertainty in the observations of the process, and the classical hypothesis testing may not be appropriate. This means that the existence of uncertainty in the observations, leads to the appearance of fuzzy data. Therefore, if we take the case of monitoring the sample mean, that sample mean is a fuzzy number ($\widetilde{m}_0$).

5.12.1. Fuzzy Acceptance Region

To deal with the above situation, Faraz et al. (2010), proposed to define a fuzzy acceptance region $\tilde{A} = (-k_a, -k_a + n, k_a - n, k_a)$, and also, to obtain the fuzzy type I and type II errors rate. Based on Zadeh's (1968) fuzzy probability, they defined the fuzzy probability of type I error and the fuzzy probability of type II error too.

5.12.1.1. Fuzzy probability of type I error

$a_{Zadeh} = P_{Zadeh}(\tilde{R}) = 1 - P_{Zadeh}(\tilde{A})$

Where,

$P_{Zadeh}(\tilde{A}) = \int_{-k_a}^{k_a} \mu_{\tilde{A}}(z) f(z) dz$

$(-k_a, k_a) \rightarrow$ The $\%100(1 - a)$ confidence interval

$f(z) \rightarrow$ The density function of the standard normal distribution

$$\mu_{\tilde{A}}(z) = \begin{cases} \frac{z+k_a}{n} & when\ -k_a \leq z \leq -k_a + n \\ 1, & when\ -k_a + n \leq z \leq k_a - n \\ \frac{k_a - z}{n} & when\ k_a - n \leq z \leq k_a \end{cases}$$

$\rightarrow$ The membership function of the fuzzy acceptance region $\tilde{A}$

$n \rightarrow$ Gives the permissible range

5.12.1.2. Fuzzy probability of type II error

$$b_{Zadeh} = P_{Zadeh}(\tilde{A}_D)$$

Where,

$P_{Zadeh}(\tilde{A}_D) = \int_{-k_a+d}^{k_a+d} \mu_{\tilde{A}_D}(z) f(z) dz$

$\tilde{A}_D = \tilde{A} + d$

$d = \frac{m_0 - m_1}{\sigma/\sqrt{n}}$

$\mu_{\tilde{A}_D}(z) \rightarrow$ The membership function of the fuzzy region $\tilde{A}_D$

(when $d = 0$, then $b_{Zadeh} = 1 - a_{Zadeh}$)

Faraz et al. (2010), introduced a fuzzy control chart for monitoring variables when uncertainty and randomness are combined. Also, they showed that when the Shewhart control charts are used, the control limits must be adjusted to enhance the existing fuzziness in the process mean.

5.12.2. Construction of fuzzy Shewhart control charts

The proposed fuzzy control charts are based on a fuzzy acceptance region $\tilde{A}$. After specifying the value of crisp statistic z, then they compared it with the fuzzy acceptance region $\tilde{A}$ and after that obtained the fuzzy set $\tilde{Q}$ as follows:

$$\tilde{Q} = \left\{\frac{\xi}{\tilde{A}}, \frac{1-\xi}{\tilde{R}}\right\}$$

Where,

$\tilde{A} \rightarrow$ Fuzzy acceptance region

$\tilde{R} \rightarrow$ Fuzzy rejection region

$\xi \rightarrow$ The grade of belonging the value z to the fuzzy acceptance region $\tilde{A}$, when the process possibility being in – control

$1 - \xi \rightarrow$ The grade of belonging the value z to the fuzzy acceptance region $\tilde{A}$, when the process possibility being out – of – control

(when $0 < \xi < 1$, is not easy to judge if the process is in – control of out – of – control)

In order to retain the standard format of Shewhart control charts, to facilitate the plotting of sample statistic on the chart, and to facilitating the decision – making, it is necessary to convert the fuzzy set $\tilde{Q}$, into its representative value for in – control and out – of – control cases. At this point, Faraz et al. (2010), introduced the usage of defuzzification methods based on heuristic ideas. Some of those defuzzification methods are the maximum defuzzifier operator (which corresponds to the maximum membership value), and the fuzzy set central tendency defuzzifier operator (the fuzzy median and the fuzzy average).

The adjusted control limits for the fuzzy Shewhart charts for monitoring the process mean are:

$$\begin{cases} UCL = m_0 + (\kappa_\alpha - C_\xi \cdot n)\,{}^{\sigma}/_{\sqrt{n}} \\ CL = \mu_0 \\ LCL = m_0 - (\kappa_\alpha - C_\xi \cdot n)\,{}^{\sigma}/_{\sqrt{n}} \end{cases}$$

Where,

$C_\xi \rightarrow$ The minimum grade of acceptance and chosen based on ξ that corresponds to the defuzzification method that it used.

$$Process\ Control = \begin{cases} if\ \ \xi < C_\xi \ \Rightarrow the\ process\ is\ out - of - control \\ if\ \ \xi \geq C_\xi \Rightarrow the\ process\ is\ in - control \end{cases}$$

Finally, Faraz et al. (2010), concluded that the fuzzy control limits are tighter and more useful than the Shewhart control limits. Furthermore, these fuzzy

Shewhart control charts have the advantageous of simplicity with respect to the other fuzzy control charts because its control limits are adjusted and direct analogue of classical Shewhart control charts.

5.13. Conclusion

As we seen above analytically, several researchers have deal on fuzzy control charts pertained to uncertainty in human cognitive processes. When human subjectivity plays an important role in defining the quality characteristics, the classical crisp control charts may not be applicable since they require certain information.

Also, we have seen that many investigators, in order not to lose information about the process, decided not to use any defuzzification methods to convert the fuzzy observations into its representative values.

Using the traditional Shewhart control charts, the judgment we can get are in the form of binary classification as *"Either the process is in – control or the process is out – of – control"*. As we have seen above, using fuzzy control charts, it is able to handle several intermediate decisions about the process.

Furthermore, with fuzzy control charts, a more flexible and informative evaluation of the considered process can be made.

Finally, below are presented some advantages and disadvantages of fuzzy control charts. The usage of fuzzy control charts provides more accurate control standards for the process based on expert's experience expressed in degree of membership. Also, those control charts are more flexible for the definitions of the fuzzy inference rules. In the other hand, the inference outcomes are based on the subjectivity experience rules and also supplemental rules for systematic changes of the traditional control charts cannot be used.

In future researches, we would except to develop further rules for detecting the variation of process in order to increase the sensitivity of the fuzzy control charts to small process shifts or to other unusual patterns such that, users may respond more rabidly to special causes. Also, the procedures used to develop the fuzzy control intervals for both variables and attributes can be simply extended to cover the cases of non – symmetric fuzzy numbers as process parameters.

CHAPTER 6

FUZZY MULTIVARIATE CONTROL CHARTS

6.1. Introduction

When products are classified into mutually exclusive linguistic categories, fuzzy control charts are used. As we have seen analytically in Chapter 4, in order to monitor a single process of a variable quality characteristic, different procedures are proposed by various inspectors, were introduced and discussed the construction of these charts.

Specifically, in many practical and realistic cases, the binary classification of a product does not change abruptly from satisfactory to worthless. Thus, arises the need for intermediate levels to describe the product quality and the QCs that cannot be expressed numerically are associated with linguistic terms referred to as "very good", "good", "medium", "poor", etc, as introduced by Wang & Raz (1990).

Firstly, Wang & Raz (1990) developed fuzzy control charts for linguistic data which are mainly based on membership and probabilistic approaches. After that, Kanagawa et al. (1993) proposed an assessment of intermediate quality levels instead of the traditional binary judgment. Also, Taleb & Limam (2002) discussed the construction of fuzzy control charts based on fuzzy and probability theory, and contrary to the conclusion of Wang & Raz (1990), they concluded that the choice of degree of fuzziness affected the sensitivity of control charts. And finally, Gulbay et al. (2004) proposed an $a-$ level fuzzy control chart for attributes in order to reflect the vagueness of data and tightness of inspection. Furthermore, Laviolette et al. (1995) compared fuzzy and probability approach for construction of control charts for linguistic data, and suggested the superiority of the probability approach based on a simpler computational implementation.

The above occur in the case of monitoring only one quality characteristic ($p = 1$), where a univariate control chart is required to control a multinomial process. In the case of monitoring more than one quality characteristics, simultaneously, when $p = 2,3,4, \ldots$, multivariate control charts is introduced.

Multivariate quality control methods overcome the disadvantage appear in univariate control methods, where for a single process, many variables may be monitored and even controlled by monitoring several variables simultaneously. Also, in many production processes, multivariate QCs tend to be correlated and therefore results could be misleading and difficult to interpret.

In the case of categorical variables, such as sex, race, age group or educational level, Woodall et al. (1997) reviewed the procedures for monitoring multinomial process when items are classified into distinct categories (groups).

This chapter is an extension of Chapter 4, where we are going to look at the monitoring of several quality characteristics, simultaneously, using fuzzy multivariate control charts.

6.2 Fuzzy Multinomial Control Charts

When the data is presented in linguistic form, in the case of monitoring $p = 1$ multinomial quality characteristic, a univariate fuzzy control chart is used to control such process.

When quality control for variables is not feasible, linguistic data provides more information than the binary classification. Amirzadeh et al. (2008), proposed the construction of fuzzy multinomial control chart ($FM-$ chart), following the steps below:

1. Assume that $\tilde{L} = \{(C_1, L(C_1)), \ldots, (C_w, L(C_w))\}$ is a linguistic variable

2. Suppose that p_i is the probability that an item is C_i $(i = 1, \dots, w)$

3. Assume that a random sample of $n-$ items is selected

4. Let X_i $(i = 1, \dots w)$ be the number of items that are C_i $(i = 1, \dots, w)$

5. $(X_1, X_2, \dots, X_w)$ has a multinomial distribution with parameters n and $p_1, p_2, \dots, p_w$. Each $X_i \sim$ binomial distribution with mean np_i and variance $np_i(1-p_i)$

6. The weighted average of $\tilde{L}(l_i)$ is:

$$\bar{\tilde{L}} = \frac{1}{\sum_{i=1}^{w} X_i} \sum_{i=1}^{w} X_i \tilde{L}(C_i) = \frac{1}{n} \sum_{i=1}^{w} X_i \tilde{L}(C_i)$$

where $i = 1, \dots, w$

7. The control limits for the $FM-$ chart is:

$$\begin{cases} UCL = \mathbb{E}\left(\bar{\tilde{L}}\right) + k\sqrt{\mathbb{V}ar\left(\bar{\tilde{L}}\right)} \\ CL = \mathbb{E}\left(\bar{\tilde{L}}\right) \\ UCL = \mathbb{E}\left(\bar{\tilde{L}}\right) - k\sqrt{\mathbb{V}ar\left(\bar{\tilde{L}}\right)} \end{cases}$$

Based on a theorem [Amirzadeh et al. (2008) p. 28], they calculated the mean and variance of the weighted average of $\tilde{L}(C_i)$ *where* $i = 1, \dots w$, $\bar{\tilde{L}}$:

$$\mathbb{E}\left(\bar{\tilde{L}}\right) = \sum_{i=1}^{w} p_i \tilde{L}(C_i)$$

$$\mathbb{V}ar\left(\bar{\tilde{L}}\right) = \frac{1}{n}\left[\sum_{i=1}^{w} p_i (1-p_i) \tilde{L}^2(C_i) - 2\sum_{i=1}^{w} \sum_{j=1, i<j}^{w} p_i p_j \tilde{L}(C_i) \tilde{L}(C_j)\right]$$

Then, Amirzadeh et al. (2008), using an illustrative example from a production process, they compared $FM-$ control chart with the traditional $P-$ chart and concluded that $FM-$ chart leads to better results than the $P-$ chart, if the number of categories and their degrees of membership are well selected.

6.3. Multivariate Control Charts

On the other hand, in the case of monitoring more than one multinomial quality characteristics, two approaches are suggested in the papers of Taleb & Limam (2005), Taleb et al. (2006), and Alipour & Noorossana (2010). When the data is presented in linguistic form, they suggested the construction of control charts to monitor multivariate attribute processes. The construction of the proposed control charts is analyzed using fuzzy set theory and probability theory.

6.3.1. Fuzzy Multivariate Control Charts

6.3.1.1. The main idea of fuzzy multivariate control charts

The main idea of fuzzy multivariate control charts follow presented analytically:

Suppose that there are $p-$ related quality characteristics (QCs):

$$C_1, C_2, \dots, C_p$$
$$[where\ C_j: for\ j = 1,2,\dots,p]$$

which are controlled jointly. Each QC, C_j, is characterized by q_j (categories) linguistic terms which are described by fuzzy term set $T(C_j)$:

$$\begin{cases} T(C_1) = \{C_{11}, C_{12}, \dots, C_{1q_1}\} \\ T(C_2) = \{C_{21}, C_{22}, \dots, C_{2q_2}\} \\ \vdots \\ T(C_{p-1}) = \{C_{(p-1)1}, C_{(p-1)2}, \dots, C_{(p-1)q_{(p-1)}}\} \\ T(C_p) = \{C_{p1}, C_{p2}, \dots, C_{pq_p}\} \end{cases}$$

Also, each term C_{jh} in term set $T(C_j)$ is characterized by a membership function $\mu_{jh}(x)$, where x is the measure of quality level and $x \in [0,1]$.

Let F_{jh} be the fuzzy set associated with each linguistic variable C_{jh}. Thus $\mu_{jh}(x)$ is the membership function of the fuzzy set F_{jh}.

Firstly, assume that there is a sample A from $n-$ observations expressed as:

$$A = \{[(F_{11}, n_{11}), \dots (F_{1q_1}, n_{1q_1})], \dots, [(F_{p1}, n_{p1}), \dots, (F_{pq_p}, n_{pq_p})]\}$$

Where,

$n_{jh} \rightarrow$ The number of observations classified by linguistic variable C_{jh}

Then, each quality characteristic C_j is associated with only one fuzzy subset:

$$F_j = \frac{1}{n} \sum_{h=1}^{q_j} n_{jh} F_{jh}$$

Thus, a sample A becomes:

$$A = \{F_1, F_2, \dots, F_p\}$$

In addition, each fuzzy subset F_j is converted into its representative value R_j using one of the four transformation methods introduced by Wang & Raz (1990), [fuzzy median, fuzzy median, fuzzy average, $\alpha-$ level fuzzy midrange], and using a triangular fuzzy number (a_{1j}, a_{2j}, a_{3j}). The representative value of F_j (when fuzzy median transformation is used) is:

$$R_j = \begin{cases} a_{3j} - \sqrt{\frac{(a_{3j}-a_{1j})(a_{3j}-a_{2j})}{2}}, for\ a_{2j} < \frac{a_{3j}+a_{1j}}{2} \\ a_{1j} - \sqrt{\frac{(a_{3j}-a_{1j})(a_{2j}-a_{1j})}{2}}, for\ a_{2j} > \frac{a_{3j}+a_{1j}}{2} \end{cases}$$

Finally, a sample A is now expressed as:

$$R_A = (R_{A1}, \dots, R_{Ap})'$$

When there are $k-$ samples each of $n-$ observations, then:

$$R = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1p} \\ \vdots & \ddots & \ddots & \vdots \\ R_{k1} & R_{k2} & \cdots & R_{kp} \end{pmatrix}$$

Where,

$R_{ij} \rightarrow$ The representative value of fuzzy number F_j in the sample i

6.3.1.2. T_f^2 statistic

Having the Hotteling's T^2 statistic:

$$T^2 = (R-m)'\Sigma^{-1}(R-m)$$

Where,

$m' = [m_1, \dots m_p] \rightarrow$ The vector of in – control means for each QC

$\Sigma \rightarrow$ The covariance matrix of QCs

And when the process is in – control, it should estimate the parameters m and Σ. The estimation value of m is:

$$\overline{R} = (\overline{R}_1, \dots \overline{R}_p)$$

Where,

$\overline{R}_j = \frac{1}{k}\sum_{i=1}^{k} R_{ij}$

$R_{ij} \rightarrow$ The representative value of the fuzzy subset associated with the i^{th} sample on the j^{th} QC

The estimation of Σ is:

$$S = \begin{pmatrix} S_1^2 & S_{12} & \cdots & S_{1p} \\ S_{21} & S_2^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ S_{p1} & S_{p2} & \cdots & S_p^2 \end{pmatrix}$$

Where,

$S_j^2 = \frac{1}{k-1}\sum_{i=1}^{k}\left(R_{ij} - \overline{R}_j\right)^2 \rightarrow$ The variance of representative values

$S_{ij} = \frac{1}{k-1}\sum_{i=1}^{k}\left(R_{ij} - \overline{R}_j\right)\left(R_{ih} - \overline{R}_h\right), for\ j \neq h \rightarrow$ The covariance between the j^{th} QC and h^{th} QC

If m and Σ are estimated by $\overline{R}$ and S respectively, the representative value of the Hotteling's T^2 statistic is:

$$T_f^2 = \left(R - \overline{R}\right)' S^{-1}\left(R - \overline{R}\right)$$

Contrary with Hotteling's T^2 distribution, the distribution of T_f^2 statistic is difficult to determine directly, and have to rely on asymptotic theory to estimate that, using the bootstrap resampling method. This method can estimate the distribution and the control limits of the T_f^2 statistic, following the steps below:

1. Calculate the $\overline{R}$ and S^{-1} from available empirical observations
2. Draw with replacement, from the observation data, B new samples of the same size

3. Compute the statistic $T_{f_i}^2 = \left(R - \overline{R}\right)' S^{-1}\left(R - \overline{R}\right)$ for each new sample i, $(i = 1,2, \dots B)$
4. Set the UCL such as the false alarm rate will be equal to a predefined value

Then, the plotted statistic T_f^2 is obtained after transforming the fuzzy observations into their representative values, and its distribution is derived using Bootstrap resampling method.

This approach is based on Wang & Raz (1990) approach and its goal is to determine a statistic which depends on a combination of all quality characteristics. Similar to the Hotteling's T^2 statistic, the proposed statistic:

$$T_f^2 \sim \frac{(m+n)(m-1)}{mn(m-p)} F_{p,m-p}$$

Where,
$m \rightarrow$ The mean vector corresponding to sample i of $n-$ observations
$S \rightarrow$ The covariance matrix
$F_{p,m-p} \rightarrow$ The F distribution with p and $m-p$ degrees of freedom

After that, the UCL of the fuzzy multivariate control chart, is chosen to be a precise percentile of $F_{p,m-p}$ distribution and the process control is obtained as:

$$Process\ Control = \begin{cases} T_f^2 > UCL \implies process\ out-of-control \\ \quad otherwise \implies process\ in-control \end{cases}$$

6.3.1.3. Interpretation of out – of – control signals

As Taleb et al. (2006) mentioned the interpretation of out – of – control signals is based on response to the question: "Which one of the $p-$ QCs is responsible for the signal?". In fact, when an out – of – control signal is generated by the multivariate fuzzy chart, the values d_j are computed for $j = 1,2, \dots p$:

$$d_j = T_f^2 - T_{f_i}^2$$

Where,

$T_{f_i}^2 \rightarrow$ The value of the statistic T_f^2 for all QCs except the j^{th} one

$T_f^2 \rightarrow$ The value of the statistic T_f^2 for all QCs

Therefore, the attribute quality characteristic with the largest value of d_j is the most responsible for the signal.

6.3.2. Multivariate Probability Control Chart

In order to construct a multivariate control chart for monitoring multinomial processes, they proposed the Z_{ij}^2 statistic:

$$Z_{ij}^2 = \sum_{h=1}^{q_j} \frac{\left(n_{ijh} - n_{0jh}\right)^2}{n_{ijh} + n_{0jh}}$$

This statistic, test the homogeneity of proportions between the base period 0 when the process is assumed to be in – control and each period i ($i = 1,2, \dots, m$)

Where,

$n_{ijh} \rightarrow$ The number of units classified by quality characteristic j into category h in the period i

$n_{0jh} \rightarrow$ The number of units classified by quality characteristic j into category h in the period 0

Also $m = \sum_{j=1}^{p} m_j$, where, $m_j = \sum_{i=1}^{m} Z_{ij}^2$ with $(q_j - 1)$ degree of freedom. Using Satterthwaite's (1946) approximation, the degree of freedom of m is given by the following type:

$$v = \frac{m^2}{\sum_{j=1}^{p}\left(m_j\right)^2/(q_j-1)}$$

6.3.2.1. W^2 statistic

Using a combination of chi – squared statistics, where its distribution is derived from Satterthwaite's (1946) approximation, they proposed a statistic W_i^2 for monitoring multivariate processes for multinomial attributes based on the probability theory, and constructed a fuzzy multivariate control chart.

The proposed statistic is:

$$W_i^2 = \sum_{j=1}^{p} Z_{ij}^2 \sim \mathcal{X}^2(v)$$

Where, the UCL of fuzzy multivariate control chart is chosen to be a percentile of $\mathcal{X}^2(v)$ distribution with an approximated degrees of freedom v. The process control is obtained as:

$$Process\ Control = \begin{cases} W_i^2\ out\ of\ control\ limits\ \Rightarrow process\ out-of-control \\ \qquad otherwise \Rightarrow process\ in-control \end{cases}$$

6.3.2.2. Interpretation of out – of – control signals

One of the most important steps of multivariate attribute control charts is the interpretation of out – of – control signals. When the control chart declares an out – of – control signal, the following steps are needed to identify which attribute is more responsible:

1. Compute m and v using Satterthwaite's (1946) approximation for the combination of all quality characteristics but the j^{th} one. Then for a certain quality characteristic c_t, m and v values are respectively:

$$m_t = \begin{cases} \sum_{j=1}^{p} m_j \ for\ j \neq t \\ v_t = \frac{m_t^2}{\sum_{j=1}^{p} (m_j)^2 / (q_j - 1)} \ for\ j \neq t \end{cases}$$

2. Let W_{it}^2 be the computed value of W_i^2 without considering the j^{th} quality characteristic. Then the statistic $W_{it}^2 = \sum_{j=1}^{p} Z_{ij}^2$, for $j \neq t$ has a $\mathcal{X}^2(v_t)$ distribution

3. The UCL_t for each statistic W_{it}^2 is taken to be a percentile of the $\mathcal{X}^2(v_t)$ distribution with $t = 1,2, \ldots p$

4. Compute the value of $d_t = W_{it}^2 - UCL_t$, for $t = 1,2, \ldots p$

5. Plot univariate control charts for each quality characteristic

6.3.3. Disadvantages of the proposed multivariate control charts

Furthermore, Taleb & Limam (2005) using an illustrative example from frozen food, they constructed the $MFQCC$ (Multivariate Fuzzy Quality Control Chart) and $MAQCC$ (Multivariate Attribute Quality Control Chart), and finally they denoted the above disadvantages:

1. $MFQCC$: As mentioned above, the $MFQCC$ is based on fuzzy theory, and thus is strongly related to the choice of membership function and the degree of fuzziness.

2. $MAQCC$: Also, the distribution of the statistic used by $MAQCC$, cannot be determined directly and it is derived from Satterthwaite's (1964) approximation.

6.4. Fuzzy $MEWMA$ Control Chart

Considering the $MEWMA$ control chart, Alipour & Noorossana (2010) proposed the construction of fuzzy $MEWMA$ control chart.

Suppose that $R_i\,, (i = 1,2, \dots, p)$ is the representative value corresponding to the fuzzy number F_i, such that:

$$R_i \sim N_p(m_i, \Sigma_F)$$

Where,

$m_i \rightarrow$ The mean vector

$\Sigma_F \rightarrow$ The known fuzzy covariance matrix

Without loss of generality, when the process is in – control, they assumed that $m_i = (0,0, \dots 0)'$.

Then,

$$Z_i = \lambda R_i + (1 - \lambda) Z_{i-1}\,, i = 1,2, \dots$$

Where,

$\lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_p \end{pmatrix} \rightarrow$ The diagonal matrix with $0 < \lambda_i \leq 1$

$Z_0 = 0$

$Z_i \rightarrow$ The fuzzy $MEWMA$ vector

The fuzzy $MEWMA$ control chart is a plot of a statistic:

$$F_i^2 = Z_i{}' \Sigma_{Z_i}{}^{-1} Z_i$$

Where,

$\Sigma_{Z_i} \rightarrow$ The covariance matrix of fuzzy $MEWMA$ vectors

The process control conditions for the fuzzy $MEWMA$ control chart are:

$$Process\ Control = \begin{cases} process\ out - of - control, & if\ F_i^2 \geq h_F \\ process\ in - control, & otherwise \end{cases}$$

Where,
$h_F \rightarrow$ The predefined threshold yielding the desired in – control ARL

Also, Alipour & Noorossana (2010) referred to the special case of $\lambda_i = \lambda$, for $i = 1,2, \dots p$.

Alipour & Noorossana (2010), using an example, compared the performance of fuzzy $MEWMA$ control chart and existing fuzzy Hotteling's T^2 control chart, using the ARL criterion which computed based on bootstrap resampling data. They concluded that the fuzzy $MEWMA$ chart indicates a superior performance over the fuzzy Hotteling's T^2 chart.

6.5. Conclusion

Traditional multivariate control charts are designed to monitor vectors of variable or attribute quality characteristics. However, there are certain situations where data are expressed in linguistic terms, which can be used more effectively, and may be a realistic choice for monitoring the quality of a product or process.

As we have seen, the fuzzy multivariate control chart is an alternative control chart for handling linguistic observations, when more than one quality characteristic is monitored, simultaneously.

Also, fuzzy multivariate control charts indicate superior performance, compared to the performance of a combination of univariate control charts in order to monitor multivariate attribute processes, when data is in the linguistic form.

Finally, some other existing multivariate control charts such as $EWMA$ and $CUSUM$ charts can be generated and developed to monitor multivariate process for multinomial categorical data and may be compared to the charts that suggested in the papers of Taleb & Limam (2005), Taleb et al. (2006), and Alipour & Noorossana (2010).

CHAPTER 7

CONCLUSION

As we have seen analytically, several researchers have deal on fuzzy control charts pertained to uncertainty in human cognitive processes. When human subjectivity plays an important role in defining the quality characteristics, the classical crisp control charts may not be applicable since they require certain information.

The usage of intermediate levels (linguistic terms) in order to describe the quality of products provides more information about the process, so the ability of the control chart to detect a process shift increases.

In this thesis, we have seen briefly that Bradshaw (1983) introduced the fuzzy control chart concepts, when Wang & Raz (1990) and Raz & Wang (1990) introduced linguistic variables to assess the product quality and proposed the membership and probabilistic approaches, in order to monitor the process average.

After that, Kanagawa et al. (1993), proposed the construction of fuzzy control charts to monitor the process average, as well as the process variability too. Gulbay et al. (2004) using $a-$ cuts, they proposed the construction of fuzzy control charts for linguistic data. In addition, fuzzy multinomial control charts proposed by Taleb & Limam (2005).

Furthermore, Taleb & Limam (2002) proposed the construction of fuzzy and probabilistic control charts, when Cheng (2005) proposed the construction of control charts using fuzzy numbers for fuzzy process control. Additionally, Gulbay & Kahraman (2007) constructed fuzzy control limits using the direct fuzzy approach in order to prevent the loss of information included by the fuzzy sample.

Also, we have seen that, the fuzzy multivariate control chart is an alternative control chart for handling linguistic observations, when more than one quality characteristics is monitored, simultaneously. Fuzzy multivariate control charts indicate superior performance compared to the performance of a combination o univariate control charts in order to monitor multivariate processes, when data is in the linguistic form.

We have also seen in detail that and other researchers have offered in the construction and interpretation of fuzzy control charts, with the urge to help solving problems created by the presence of fuzzy observations. They concluded that, using fuzzy control charts can provide a more flexible and informative evaluation of the considered process.

Representing the linguistic variables as fuzzy sets, retains the ambiguity and vagueness inherent in natural languages and improves the expressive ability of quality assurance inspectors. But, how many linguistic terms should be defined or how should the degree of membership of linguistic terms be constructed, may will be the subjects of future research.

The survey in the area of fuzzy control charting methodology is still open, and taking into consideration the continuous development of science and technology, a lot of work has to done in the future.

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